Warm-up Problems

- Compute the homology groups of S^n for all $n \geq 0$.
- (Jan 2006, B9) Use Mayer-Vietoris sequences to compute the singular homology groups of the 2-torus $T = S^1 \times S^1$. (You may use knowledge of the homology groups of the circle S^1 in your calculations.)

1. (June 2005, B7) Find a Δ-complex structure on the space X obtained by identifying three distinct points a, b, c in the 2-sphere to a point, and compute the simplicial homology groups of X.
2. (Jan 2006, B8) Find a Δ-complex structure on the space X obtained by identifying the boundaries of three copies of the unit disk together (using identity maps), and compute the simplicial homology groups of X.
3. (June 2007, B4) Find the Euler characteristic of $X = (\Delta^5)^{(2)}$, the 2-skeleton of the 5-simplex. Show that $H_1(X) = 0$, $H_2(X)$ is free abelian, and compute the rank of $H_2(X)$.
4. (June 2008, B7) Let Y be a Δ-complex.
   (a) Prove that if $H_4(Y) \neq 0$, then Y must have a simplex of dimension 4.
   (b) Prove that if $H_4(Y) = \mathbb{Z}/7\mathbb{Z}$, then Y also must have a simplex of dimension 5.
5. (June 2008, B8) Let X be a hexagon in $\mathbb{R}^2$. Define an equivalence relation on X corresponding to labeling the 6 edges in the boundary of X in counterclockwise fashion in order by : counterclockwise a, counterclockwise b, counterclockwise a, counterclockwise b, counterclockwise a, counterclockwise b. Let M be the corresponding quotient space. Computer $H_n(M)$ for all $n \geq 0$.
6. (June 2009, B8) Let X be the space obtained by attaching a torus $T = S^1 \times S^1$ to a cylinder $C = S^1 \times I$ via a homeomorphism of the circle $S^1 \times \{(1, 0)\}$ of T with the circle $S^1 \times \{0\}$ of C. Compute the homology groups of X.
7. (June 2010, B8) Let X be the space obtained from the 2-simplex Δ^2 by identifying all three vertices together. Describe a Δ-complex structure on the space X and compute its homology groups.
8. (June 2014, B8) Let X be a path-connected Hausdorff space. The *suspension* SX of X is the quotient of the space $X \times I$ by the smallest equivalence relation such that $(p, 0) \sim (q, 0)$ and $(p, 1) \sim (q, 1)$ for all $p, q \in X$. Prove that the homology groups satisfy $H_i(SX) = H_{i-1}(X)$ for all $i > 1$.
9. (May 2015, B7) Find the (simplicial) homology groups of the space X obtained from the 3-simplex $\Delta^3 = [v_0, v_1, v_2, v_3]$ by gluing the face $[v_0, v_1, v_2]$ to the face $[v_0, v_2, v_3]$ and the face $[v_0, v_1, v_3]$ to the face $[v_1, v_2, v_3]$ (respecting the ordering of the vertices as written (i.e., the first gluing map sends v_1 to v_2, and so on)).
10. (May 2015, B8) Compute the (reduced) singular homology groups of the space $X = S^1 \times (S^1 \vee S^1)$, which can be thought of as two copies of $S^1 \times S^1$ glued together along their copies of $S^1 \times \{x_0\}$. [You may use your knowledge of the homology groups of $T^2 = S^1 \times S^1$ in your calculations.]