Group Theory Qual Review

Robert Won
Prof. Rogalski

1 (Some) qual problems and (some) techniques

- (Spring 2008, 1) Let G be a finite group and H a proper subgroup. Show that G is not the set-theoretic union of the conjugates of H.

  Consider the intersection and count.

- (Spring 2008, 2) Classify all groups with 99 elements.

  These types of problems are very common, so do a lot of these as practice. Your tools include Sylow, semidirect products, etc.

- (Spring 2008, 3) Let p be prime. If $|G| = p^n$ and N is a normal subgroup, show that N intersects the center of G nontrivially.

  A normal subgroup is a union of conjugacy classes. Count.

- (Spring 2007, 1) Let p be a prime and G a group of order p^3.

  (a) Prove that G has a normal subgroup of order p^2.

  (b) Assume that G has a cyclic normal subgroup N of order p^2 generated by some element n. Let g be an element not in N.

     i. If the order $|g|$ of g is p^3, classify the possible G up to isomorphism.

     ii. If the order $|g|$ of g is p, classify the possible G up to isomorphism

  Use Sylow, semidirect products.

- (Fall 2007, 1) Let G be a group of order $240 = 2^4 \cdot 3 \cdot 5$.

  (a) How many p-Sylow subgroups might G have, for $p = 2, 3, 5$?

  (b) If G has a subgroup of order 15, show that it has an element of order 15.

  (c) Say G does not have a subgroup of order 15. Show that the number of 3-Sylows is 10 or 40.

  Use Sylow, use Sylow again on the subgroup of order 15, semidirect products.

- (Fall 2006, 2.1) Let p be a prime number. $(\mathbb{Z}/p^2\mathbb{Z})^\times$ denotes the multiplicative group consisting of all congruence classes $\hat{x} \in \mathbb{Z}/p^2\mathbb{Z}$ such that $\gcd(x, p) = 1$.

  (a) Show that the order of $1 \hat{+} p$ in $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is equal to p.

(b) Use (a) to construct a non-abelian group of order p^3.

(c) Describe the non-abelian group in (b) via generators and relations.

Semidirect products, etc.

- (Fall 2006, 2.2) Let G be a group. Let $r \geq 2$ be an integer. Assume that G contains a non-trivial subgroup H of index $[G:H] = r$. Prove the following.

  (a) If G is simple, then G is finite and $|G|$ divides $r!$.

  (b) If $r \in \{2, 3, 4\}$, then G cannot be simple.

  (c) For all integers $r \geq 5$, there exist simple groups G which contain non-trivial subgroups H of index $[G:H] = r$.

  If G is simple, act on cosets of H by multiplication to give an injection $G \to S_n$. This is a common technique when you are dealing with simple groups. Also see Dummit and Foote pp. 201-213.

2 (Some) group things to know

- Basic facts and definitions. (homomorphisms, isomorphism theorems, subgroups, normal subgroups, normalizers, centralizers, quotient groups, cyclic groups, dihedral groups, symmetric groups, etc.)

- $H \leq G$. Given $a, b \in G$, either $aH = bH \Leftrightarrow a^{-1}b \in H$ or $aH \cap bH = \emptyset$. So cosets partition G and $|aH| = |H|$.

- $H \trianglelefteq G$. Then $|G/H| = |G|/|H| = [G:H]$.

- $K \leq H$, $H \leq G$. Then $[G:K] = [G:H][H:K]$.

- The kernel of a group homomorphism is a normal subgroup.

- G act on A, then for each $g \in G$, we get $\sigma_g : A \to A$. This σ_g is a permutation of A and the map $G \to S_a$, $g \mapsto \sigma_g$ is a homomorphism.

- (Orbit-stabilizer) $|\mathcal{O}_x| = [G : G_x] = |G|/|G_x|$.

- Automorphisms

  If $H \trianglelefteq G$, then G acts by conjugation on H as automorphisms of H. Also $G/C_G(H) \cong$ a subgroup of $\mathrm{Aut}(H)$.

  For any $H \leq G$, $N_G(H)/C_G(H) \cong$ a subgroup of $\mathrm{Aut}(H)$.

  $G/Z(G) \cong$ subgroup of $\mathrm{Aut}(G)$.

  p a prime $\Longrightarrow \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$.

- Isomorphism Theorems

  First Isomorphism Theorem: If $\varphi : G \to H$ is a homomorphism, then $\ker \varphi \trianglelefteq G$ and $G/\ker \varphi \cong \varphi(G)$.

φ injective $\iff$ ker $\varphi = 1$

Second Isomorphism Theorem: $A \leq G$, $B \leq G$ and $A \leq N_G(B)$ (or $B \trianglelefteq G$). Then $AB \leq G$ and $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

$|AB| = |A||B|/|A \cap B|$.

Third Isomorphism Theorem: $H \trianglelefteq G$ and $K \trianglelefteq G$ with $H \leq K$. Then $K/H \trianglelefteq G/H$ and $\overline{G}/\overline{K} \cong G/K$.

- Characteristic subgroups

  Characteristic subgroups are normal.

  If $H \leq G$ is the unique subgroup of a given order, then H char G.

  K char H and $H \trianglelefteq G \Longrightarrow K \trianglelefteq G$.

- (Lagrange's Theorem) G a finite group, $H \leq G$, then $|H| \mid |G|$.

- (Cauchy's Theorem) G a finite group and p a prime such that $p \mid |G|$ then G has an element of order p.

- (Sylow's Theorem)

  Sylow p-subgroups of G exist.

  If $P \in Syl_p(G)$ and Q any p-subgroup of G, then $Q \leq gPg^{-1}$.

  $n_p \equiv 1 \pmod p$ and $n_p = [G : N_G(P)]$.

  $n_p = 1 \iff P \trianglelefteq G \iff P$ char $G \iff$ All subgroups generated by elements of p-power order are p-groups.

- (Fundamental Theorem of Finitely Generated Abelian Groups)

  $G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$

  Invariant factors: $n_i \mid n_{i+1} for 1 \leq i \leq s-1$

  Elementary divisors

  If n is the product of distinct primes, the only abelian group of order n is the cyclic group of order n, $\mathbb{Z}_n$.

  $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \iff (m, n) = 1$.

- (Class equation)

  $|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$ (one x_i from each conjugacy class).

- Commutators

  $[x, y] = x^{-1}y^{-1}xy$ is called the commutator ($= 1$ iff x and y commute).

  $G' = \langle [x, y] \mid x, y \in G \rangle$ is the commutator subgroup ($= 1$ iff G abelian).

  $xy = yx[x, y]$.

  $H \trianglelefteq G$ iff $[H, G] \leq H$.

  G' char G and G/G' is abelian (the largest abelian quotient).

  If $G' \leq H$,$H \trianglelefteq G$, then G/H is abelian

- Direct products

  If $H, K \trianglelefteq G$ and $H \cap K = 1$, then $HK \cong H \times K$.

- Semidirect products

  Let K, H be groups $\varphi : K \to \mathrm{Aut}(H)$ a homomorphism. If $\sigma : K \to K$ is an automorphism of K then
  $$H \rtimes_{\varphi} K \cong H \rtimes_{\varphi \circ \sigma} K.$$

- p-groups

  $|P| = p^a$, p a prime, then:

  1. The center of p is non-trivial:
  2. $H \trianglelefteq P$ then $H \cap Z(P) \neq 1$. So every normal subgroup of order p is contained in the center.
  3. $H < P$ then $H < N_P(H)$
  4. Every maximal subgroup of P is of index p and is normal in P.

- Upper central series

  $Z_0(G) = 1$, $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ (so $Z_{i+1}(G)$ is the preimage in G of the center of $G/Z_i(G)$ under the natural projection).

  $Z_i(G)$ char G.

- Nilpotent groups

  G is nilpotent if $Z_n(G) = G$ for some n. (So abelian groups are nilpotent).

  If $|P| = p^a$ for prime a, then P is nilpotent. (p-groups have non-trivial center).

  $|G| = p_1^{a_1} \cdots p_s^{a_s}$, and $P_i \in Syl_{p_i}(G)$. TFAE:

  1. G nilpotent;
  2. $H < G$ then $H < N_G(H)$ (normalizers grow);
  3. $P_i \trianglelefteq G$;
  4. $G \cong P_1 \times \ldots \times P_s$.

  Finite abelian group is direct product of its Sylow subgroups.

  Finite group is nilpotent iff every maximal subgroup is normal

  Subgroups and factor groups of nilpotent groups are nilpotent

- Lower central series

  $G^0 = G$, $G^i = [G, G^{i-1}]$. Then $G^0 \geq G^1 \geq \cdots$

  A group is nilpotent iff $G^n = 1$ for some n.

- Derived series (Commutator series)
  $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.
  $G^{(i)}$ char G.
  G is solvable iff $G^{(n)} = 1$ for some n.
  Nilpotent groups and subgroups of solvable groups are solvable
  If G/N and N are solvable, then G is solvable.