MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 7 SOLUTIONS

1. Let $\beta \in \mathbb{C}$.

(a) Show that for all $n = 0, 1, 2, \dots,$

$$\left(\frac{\beta^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{\beta^n e^{\beta z}}{n! z^{n+1}}\,dz.$$

Solution. Applying generalized Cauchy's integral formula, we get

$$\frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{\beta^n e^{\beta z}}{n! z^{n+1}}\,dz = \frac{\beta^n}{n!}\left[\frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{e^{\beta z}}{z^{n+1}}\,dz\right] = \frac{\beta^n}{n!} \times \frac{1}{n!}\frac{d^n}{dz^n}e^{\beta z}\bigg|_{z=0} = \left(\frac{\beta^n}{n!}\right)^2.$$

(b) Show that

$$\sum_{n=0}^{\infty}\left(\frac{\beta^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2\beta\cos\theta}\,d\theta.$$

[*Hint*: Consider power series expansion of $e^{\beta/z}$ and apply (a) on $z^{-1}e^{\beta(z+1/z)}$.]

Solution. Note that

$$e^{\beta/z} = \sum_{n=0}^{\infty}\frac{\beta^n}{n! z^n}.$$

Multiplying by $e^{\beta z}$ and dividing by z, we get

$$\frac{1}{z}e^{\beta(z+1/z)} = \sum_{n=0}^{\infty}\frac{\beta^n e^{\beta z}}{n! z^{n+1}}.$$

Integrating about $\partial D(0,1)$ and using (a), we get

$$\frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{1}{z}e^{\beta(z+1/z)}\,dz = \sum_{n=0}^{\infty}\frac{\beta^n}{n!}\left[\frac{1}{2\pi i}\int_{\partial D(0,1)}\frac{e^{\beta z}}{z^{n+1}}\,dz\right] = \sum_{n=0}^{\infty}\left(\frac{\beta^n}{n!}\right)^2$$

Evaluating the line integral about the path $z : [0, 2\pi] \to \mathbb{C}$, $z(\theta) = e^{i\theta}$ and noting that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ we get

$$\frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{1}{z}e^{\beta(z+1/z)}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} e^{2\beta\cos\theta}\,d\theta.$$

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $a \in \mathbb{R}$ be an arbitrary constant.

(a) Show that if $\operatorname{Re} f(z) \le a$ for all $z \in \mathbb{C}$, then f is constant.

(b) Show that if $\operatorname{Re} f(z) \ge a$ for all $z \in \mathbb{C}$, then f is constant.

(c) Show that if $[\operatorname{Re} f(z)]^2 \le [\operatorname{Im} f(z)]^2$ for all $z \in \mathbb{C}$, then f is constant.

(d) Show that if $[\operatorname{Re} f(z)]^2 \ge [\operatorname{Im} f(z)]^2$ for all $z \in \mathbb{C}$, then f is constant.

(e) Suppose h is another entire functions and suppose there exists an $a \in \mathbb{R}$, $a > 0$, such that $\operatorname{Re} f(z) \le a \operatorname{Re} h(z)$ for all $z \in \mathbb{C}$. Show that there exist $\alpha, \beta \in \mathbb{C}$ such that

$$f(z) = \alpha h(z) + \beta$$

for all $z \in \mathbb{C}$.

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[*Hint*: if f and g are both entire, then so are $f \circ g$ and $g \circ f$; find an appropriate g so that you may apply Liouville's theorem.]

SOLUTION. Note that e^x is a monotone increasing function on $\mathbb{R}$.

- For (a), we choose $g(z) = e^z$ and note that $|e^{f(z)}| = |e^{\operatorname{Re} f(z)} e^{i \operatorname{Im} f(z)}| = e^{\operatorname{Re} f(z)} \le e^a$.
- For (c), we choose $g(z) = e^{z^2}$ and note that $|e^{f(z)^2}| = |e^{[\operatorname{Re} f(z)]^2 - [\operatorname{Im} f(z)]^2} e^{2i \operatorname{Re} f(z) \operatorname{Im} f(z)}| = e^{[\operatorname{Re} f(z)]^2 - [\operatorname{Im} f(z)]^2} \le e^0 = 1$.

Applying Liouville's theorem then implies that $e^{f(z)}, e^{f(z)^2}$ are constant functions. To show that f must also be a constant function, we differentiate $e^{f(z)}$ to get

$$0 = (e^{f(z)})' = f'(z) e^{f(z)}.$$

Since $e^{f(z)}$ is never zero, we get $f'(z) = 0$ and so f must be a constant function in (a). The same argument shows for (c) that f^2 must be a constant function and therefore f must be a constant function (since it is continuous). (b) could be deduced from (a) and (d) could be deduced from from (c) by applying (a) and (c) to $-f$. For (e), we just apply (a) to the entire function $f - ah$, which by assumption satisfies $\operatorname{Re} f(z) - a \operatorname{Re} h(z) \le 0$ for all $z \in \mathbb{C}$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Suppose there exists $\alpha, \beta \in \mathbb{C}^\times$ such that $\alpha/\beta \notin \mathbb{R}$. Show that if f satisfies the following conditions

$$f(z + \alpha) = f(z), \quad f(z + \beta) = f(z)$$

for all $z \in \mathbb{C}$, then f is constant.

SOLUTION. Given any real number $x \in \mathbb{R}$, we will write $[x]$ for the integral part of x and $\langle x \rangle$ for the fractional part of x. For example $[-5.12] = -5$ and $\langle -5.12 \rangle = 0.12$. Note that $[x] \in \mathbb{Z}$, $\langle x \rangle \in [0, 1)$, and $x = [x] + \langle x \rangle$ for all $x \in \mathbb{R}$. The condition $\alpha/\beta \notin \mathbb{R}$ implies that α, β span $\mathbb{C}$ as a real vector space of dimension 2. In other words, any $z \in \mathbb{C}$ may be written as $z = x\alpha + y\beta$ where $x, y \in \mathbb{R}$. Observe that the two conditions given may be applied recursively to obtain

$$\begin{aligned} f(z) &= f(x\alpha + y\beta) \\ &= f(\alpha\langle x \rangle + \beta\langle y \rangle + \alpha[x] + \beta[y]) \\ &= f(\alpha\langle x \rangle + \beta\langle y \rangle). \end{aligned}$$

for any $z = x\alpha + y\beta \in \mathbb{C}$. Note that for any $z = x\alpha + y\beta \in \mathbb{C}$, $\alpha\langle x \rangle + \beta\langle y \rangle \in [0, \alpha) \times [0, \beta) \subseteq [0, \alpha] \times [0, \beta]$, ie. the closed parallelogram bounded by the line segments from 0 to α and 0 to β and this is compact, and so

$$\sup_{z \in \mathbb{C}} |f(z)| = \sup_{z \in [0,\alpha) \times [0,\beta)} |f(z)| \le \max_{z \in [0,\alpha] \times [0,\beta]} |f(z)|.$$

The last term is finite by the Extreme Value Theorem in Math **104** (since $[0, \alpha] \times [0, \beta]$ is compact and f is analytic, therefore continuous) and so f is bounded. Liouville's theorem then implies that f is constant.

(b) Suppose

$$\lim_{|z| \to \infty} \frac{f(z)}{z} = 0.$$

Show that f is a constant function.

SOLUTION. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) := \begin{cases} \dfrac{f(z) - f(0)}{z - 0} & z \neq 0, \\ f'(0) & z = 0. \end{cases}$$

By Corollary **4.4**, g is an entire function too. Now,

$$\lim_{|z|\to\infty} |g(z)| = \lim_{|z|\to\infty} \left|\frac{f(z)}{z}\right| = 0.$$

Let $\varepsilon > 0$. Then there exists $R > 0$ such that

$$|g(z)| < \varepsilon$$

for all $|z| \geq R$. In particular, $|g(z)| < \varepsilon$ for all $z \in \partial D(0, R)$. Applying maximum modulus theorem (or Corollary **4.15**), we get

$$\max_{z\in \overline{D(0,R)}} |g(z)| = \max_{z\in\partial D(0,R)} |g(z)| \leq \varepsilon.$$

Therefore $|g(z)| \leq \varepsilon$ for all $z \in \mathbb{C}$. Since ε is arbitrary, we conclude that

$$g(z) = 0$$

for all $z \in \mathbb{C}$. Hence $f(z) = f(0)$ for all $z \in \mathbb{C}$ and so f is a constant function.

(a) Find all entire functions f that satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for all $n \in \mathbb{N}$.

Solution. Note that if f is an entire function, then so is f''. In particular, $f'' + f$ is continuous and so

$$f''(0) + f(0) = \lim_{n\to\infty} \left[f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right)\right] = 0.$$

Hence $f'' + f$ is zero on a subset of $\mathbb{C}$ with limit points, namely, $\{n^{-1} \mid n \in \mathbb{N}\} \cup \{0\}$ and thus by the uniqueness theorem, $f'' + f \equiv 0$ on the whole of $\mathbb{C}$. Now since f is entire, its Taylor series expansion about 0 that converges everywhere in $\mathbb{C}$, and is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Likewise for f'', we have

$$f''(z) = \sum_{n=0}^{\infty} \frac{f^{(n+2)}(0)}{n!} z^n.$$

Hence, for all $z \in \mathbb{C}$,

$$f''(z) + f(z) = \sum_{n=0}^{\infty} \left[\frac{f^{(n+2)}(0)}{n!} + \frac{f^{(n)}(0)}{n!}\right] z^n.$$

Now since $f'' + f \equiv 0$, we must have $f^{(n+2)}(0) = -f^{(n)}(0)$ for all $n \in \mathbb{N} \cup \{0\}$, ie.

$$f(0) = -f''(0) = \cdots = (-1)^n f^{(2n)}(0) = \cdots$$
$$f'(0) = -f'''(0) = \cdots = (-1)^n f^{(2n+1)}(0) = \cdots .$$

Hence

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \\ &= f(0)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + f'(0)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= f(0)\cos z + f'(0)\sin z. \end{aligned}$$

Note that the 'splitting' of the first power series into a sum of two power series is permissible because all three series have infinite radius of convergence. Hence an entire function that satisfies the given condition must be of the form

$$f(z) = f(0)\cos z + f'(0)\sin z.$$

(b) Let $n \in \mathbb{N}$ and $n \geq 2$. Find all entire functions f that satisfy

$$f(z^n) = [f(z)]^n$$

for all $z \in \mathbb{C}$.

Solution. By Theorem **4.3**, f has a power series expansion

$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$

with infinite radius of convergence. By the given condition

$$a_0 = f(0) = f^n(0) = a_0^n$$

and so either $a_0 = 0$ or $a_0 = e^{\frac{2p\pi i}{n-1}}$ for some $p \in \{1, \dots, n-1\}$.

Case I. Suppose $a_0 = e^{\frac{2p\pi i}{n-1}}$ for some $p \in \{1, \dots, n-1\}$ and f is non-constant. Let $k \in \mathbb{N}$ be the smallest positive number such $a_k \neq 0$. Hence

$$f(z^n) = 1 + a_k z^{kn} + \text{higher order terms}$$

and

$$[f(z)]^n = 1 + na_k z^k + \text{higher order terms}.$$

Since $f(z^n) = [f(z)]^n$, comparing coefficients tells us that $na_k = 0$ and so $a_k = 0$ — a contradiction. In other words, if $a_0 = e^{\frac{2p\pi i}{n-1}}$ for some $p \in \{1, \dots, n-1\}$, then f must be a constant function. Hence $f(z) = a_0 = e^{\frac{2p\pi i}{n-1}}$ for all $z \in \mathbb{C}$.

Case II. Suppose $a_0 = 0$ and f is non-constant. Again we let k be as above and observe that

$$f(z) = z^k[a_k + a_{k+1}z + a_{k+2}z^2 + \cdots] =: z^k g(z).$$

Note that g and f must have the same radii of convergence since

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_{n+k}|}$$

and hence g is also an entire function. Also $f(z^n) = [f(z)]^n$ implies

$$z^{nk} g(z^n) = z^{nk}[g(z)]^n$$

and so $g(z^n) = [g(z)]^n$. In other words, g satisfies the conditions of Case I. Hence g is a constant function and $g(z) = e^{\frac{2p\pi i}{n-1}}$ for some $p \in \{1, \dots, n-1\}$. Therefore $f(z) = e^{\frac{2p\pi i}{n-1}} z^k$ for all $z \in \mathbb{C}$.

Combining Cases I and II, we see that an entire function that satisfies the given condition must be of the form $f(z) = e^{\frac{2p\pi i}{n-1}} z^k$ for $k = 0, 1, 2, \dots$ and $p \in \{1, \dots, n-1\}$.

4. Let $\Omega \subseteq \mathbb{C}$ be a region. Let f be analytic on Ω and let $z_0 \in \Omega$. Suppose $f'(z_0) \neq 0$. Show that there is an $r > 0$ such that

$$\int_\Gamma \frac{f'(z_0)}{f(z) - f(z_0)}\, dz = 2\pi i$$

where $\Gamma = \partial D(z_0, r)$.

Solution. We know that there is an $R > 0$ such that f has a Taylor series expansion about z_0. So

$$f(z) = f(z_0) + a_1(z - z_0) + \sum_{n=2}^{\infty} a_n (z - z_0)^n$$

holds for all $z \in D(z_0, R)$. Now since $a_1 = f'(z_0) \neq 0$ and since f' is continuous at z_0, there is an $\delta > 0$ such that $f(z) - f(z_0) \neq 0$ for all $z \in D(z_0, \delta) \backslash \{z_0\}$ (if not, we can find a sequence $z_n \to z_0$, $z_n \neq z_0$, such that $f(z_n) - f(z_0) = 0$ for all $n \in \mathbb{N}$ — this will imply that $0 = \lim_{n\to\infty}(f(z_n) - f(z_0))/(z_n - z_0) = f'(z_0)$, a contradiction). Let $r = \min\{R, \delta\}$ and let the function $g : D(z_0, r) \to \mathbb{C}$ be defined by

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0, \\ f'(z_0) & z = z_0. \end{cases}$$

Now observe that g is analytic in $D(z_0, r)$ by a result in the lectures. Furthermore, g is non-zero on $D(z_0, r)$. Hence the function $h : D(z_0, r) \to \mathbb{C}$ defined by

$$h(z) = \frac{1}{g(z)}$$

is analytic on $D(z_0, r)$. Cauchy's integral formula applied to h yields

$$\frac{1}{2\pi i} \int_\Gamma \frac{h(z)}{z - z_0}\, dz = h(z_0)$$

but since

$$h(z) = \begin{cases} \dfrac{z - z_0}{f(z) - f(z_0)} & z \neq z_0, \\ \dfrac{1}{f'(z_0)} & z = z_0, \end{cases}$$

we get

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{f(z) - f(z_0)}\, dz = \frac{1}{f'(z_0)}$$

and thus

$$\int_\Gamma \frac{f'(z_0)}{f(z) - f(z_0)}\, dz = 2\pi i$$

as required.