Math 871 - Section 001 - Fall 2013 - Problem Sets

- PS1: 1.2cgkmpq, 2.1, 2.2, 6.2, 7.1, 7.5aei, 13.3, 13.4
  Due 9/5/13 for grading: 2.2b, 13.4ac

- PS2: PS2.1, 18.3, 18.5, PS2.2, 13.8
  - **PS2.1**: Let X and Y both be the set **R** of real numbers, let T_{ip} be the included point topology (where 0 is the "included point") on X, and let T_{fc} be the finite complement topology on Y. Determine whether or not the topological spaces (X,T_{ip}) and (Y,T_{fc}) are homeomorphic. (As always, be sure to prove your answer.)
  - **PS2.2**:
    (a) Show that for any set X, the set B:={U ⊆ X | X-U is infinite} ∪ {X} is a basis for a topology on X.
    (b) For every X the topology *T(B)* generated by the basis in part (a) is a topology we have already run into! Which ones? (Prove your answer!)

  Due 9/12/13 for grading: PS2.1, PS2.2

- PS3: 16.1, 16.3, PS3.1, 19.2, 19.10, 18.4, PS3.2
  - **PS3.1**: A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y.
    (a) Prove Exercise 16.4.
    (b) Show that if X_α has the topology T_α for each $\alpha \in I$, and $\prod_{\alpha \in I} X_\alpha$ has the product topology, then the projection map $\pi_\beta : (\prod_{\alpha \in I} X_\alpha) \to X_\beta$ is open.
  - **PS3.2**: Let X and Y both be the set **Z** of integers with the finite complement topology T_X=T_Y on **Z**. Let T_{prod} be the product topology on $X \times Y$, and let T_{fc} be the finite complement topology on $X \times Y$. Determine whether or not $T_{prod} = T_{fc}$. If a double containment fails, determine whether one or the other of the possible containments holds. (Prove your answer.)

  Due 9/19/13 for grading: 16.1, PS3.1(b), 18.4

- PS4: PS4.1, 17.3, PS4.2, 18.8 (Y=**R** with Euclidean topology), 17.8ab, 17.19ab, 17.20cf, 18.2
  - **PS4.1**: Let X be the set **R** of real numbers with the Euclidean topology T_X=T_{Eucl}, and let Y be the set **R** of real numbers with the included point topology T_Y=$T_{included}$ = {U ⊆ Y | 0 ∈ U} ∪ {∅}. Let f:X -> Y be defined by f(t)=t-1 for every real number t. Determine whether or not f is continuous (and prove your answer).
  - **PS4.2**: A map $f : X \to Y$ is said to be a **closed map** if for every closed set U of X, the set $f(U)$ is closed in Y.
    (a) Show that if f:X -> Y is a closed map and f(X) ⊆ B ⊆ Y, then $f|^B$ is a closed map.
    (b) Suppose that X_α has the topology T_α for each $\alpha \in I$, and $\prod_{\alpha \in I} X_\alpha$ has the product topology. Show that the projection map $\pi_\beta : (\prod_{\alpha \in I} X_\alpha) \to X_\beta$ may not be closed.
    (*Hint*: Use problem E17.20(f).)

  Due 9/26/13 for grading: PS4.1, 17.8b, 18.2

- PS5: 3.4, 22.1, 22.2, PS5.1, PS5.2, PS5.3, PS 5.4
  - **PS5.1**: Show that if $f : X \to Y$ and $g : Y \to Z$ are both homeomorphisms, then so are the inverse function f^{-1} and the composition $g \circ f$.
  - **PS5.2**:
    (a) Show that a composition of open maps is open.
    (b) Show that a finite product of open maps is open. Is the result true for an infinite product?
    (c) Show that if f:X -> Y is an open map and $f(X) \subseteq B \subseteq Y$, then $f|^B$ is an open map.
  - **PS5.3**: In each part a topological space X that is a subspace of Euclidean space is given, together with an equivalence relation ~ on X. Find a familiar space Y that is homeomorphic to the quotient space X/~, and prove your answer using Theorem I.
    (a) $X = \mathbf{R}^2$, and ~ is defined by [$(x_0,y_0) \sim (x_1,y_1)$ if and only if $x_0 + y_0^2 = x_1 + y_1^2$]. Prove your answer in this part using Theorem Q.
    (b) $X = \mathbf{R}^2$, and ~ is defined by [$(x_0,y_0) \sim (x_1,y_1)$ if and only if $x_0^2 + y_0^2 = x_1^2 + y_1^2$]. Prove your answer in this part using the result of E22.2b.
    (Note: You may use the fact that the square root function $\sqrt{\ }$: $[0,\infty) \to [0,\infty)$ (where each space has the Euclidean subspace topology) is continuous.)
    (c) $X = [-1,1] \subseteq \mathbf{R}$, and ~ is the smallest equivalence relation on X with $-p \sim p$ for all $p \in X$.
    (d) $X = \mathbf{R}$, and ~ is the smallest equivalence relation on X with $x \sim x+n$ for all x in **R** and all integers n.
  - **PS5.4**: Let X be an octagon in $\mathbf{R}^2$. Define an equivalence relation on X corresponding to labeling the 8 edges in the boundary of X in a counterclockwise fashion in order by: counterclockwise a, counterclockwise b, clockwise a, clockwise b, counterclockwise c, counterclockwise d, clockwise c, clockwise d. Let M be the corresponding quotient space. Build a concrete version of M out of paper or cloth (or any other 2-dimensional flexible material) to show that M is homeomorphic to the frosting on a doughnut with 2 holes.

  Due 10/3/13 for grading: 22.1, PS5.2c, PS5.3b

- PS6: 17.11, 17.12, 19.3
  Exam 1 10/8/13

- PS7: 20.3a, 21.1, PS7.1, PS7.2, PS7.3, 23.11, 24.3, 24.8
  - **PS7.1**: Let X be a metrizable space, and suppose that $p \in X$ and C is a closed subset of X that does not contain p. Show that there are disjoint open sets U and V in X with $p \in U$ and $C \subseteq V$.
  - **PS7.2**: Let X denote the ``flea and comb space'':
    $X = \{(0,1)\} \cup \{ (x,0) \mid 0 \leq x \leq 1 \} \cup \{ ((1/n),y) \mid 0 \leq y \leq 1 , n \in \mathbf{N} \}$, with the subspace topology from the Euclidean space $\mathbf{R}^2$.
    (a) Show that X - {(0,1)} is path-connected.
    (b) Show that X is connected.
    (c) Show that X is *not* path connected. (Hint 1: A path γ from (0,1) to any other point must first leave a neighborhood of (0,1). Show that the IVT says that it can't.) (Hint 2: The least upper bound property for the reals may be useful.)
    (d) Prove that X has two path components, one of which is *not* a closed subset of X.
  - **PS7.3**: For the following topological spaces, determine whether or not the space is connected or path connected, and find the connected components and path components.
    (a) The real line with the lower limit topology.

(b) The real line with the excluded point topology.
(c) The real line with the included point topology.
Due 10/24/13 for grading: PS7.1, PS7.3a, 24.8a

- PS8: 26.3, PS8.1, PS8.2, 26.5, 26.9, 30.4, 30.12 (2nd ctbl only), 31.2, 31.5, 32.1, 32.2
  - **PS8.1**: For the following topological spaces:
    - (1) Determine whether or not the space is compact. (That is, determine whether or not a race of space-faring snuffalumps measuring the temperatures at all of the points in their space must find that a maximum and a minimum temperature will be achieved!)
    - (2) Determine the largest natural number i for which the space has the separation property T_i.

    (a) The real line with the excluded point topology.
    (b) The real line with the included point topology.
    (c) The space $\mathbf{R}_{ll}$ consisting of real line with the lower limit topology.
    (d) The subspace [0,1] of $\mathbf{R}_{ll}$.
  - **PS8.2**: In each part, a topological space X built from subspaces of Euclidean space is given, together with an equivalence relation ~ on X. Find a familiar space Y that is homeomorphic to the quotient space X/~, and prove your answer.
    (a) $X = [0,1] \times [0,1] \subseteq \mathbf{R}^2$, and ~ is the smallest equivalence relation on X such that (x,0) ~ (x,1) and (0,y) ~ (1,y) for all $x,y \in [0,1]$.
    (b) $X = D_1 \cup D_2$ is the disjoint union of two closed disks $D_1 \cong D_2 \cong D^2 = \{(x,y) \mid x^2 + y^2 \leq 1\} \subseteq \mathbf{R}^2$ (where each disk has the Euclidean subspace topology, and X has the disjoint union topology) and ~ is the smallest equivalence relation on X such that (x,y) ~ (r,s) for all $(x,y) \in D_1$ and $(r,s) \in D_2$ satisfying (x,y)=(r,s) and $x^2 + y^2 = 1$.
    (c) Challenge: $X = D^2$ and ~ is the smallest equivalence relation on X such that (x,y) ~ (1,0) for all (x,y) satisfying $x^2 + y^2 = 1$.

  Due 11/7/13 for grading: 26.5, PS8.1(1)(d), 30.4, 32.2(T_4 only)

- PS9: PS9.1:Urysohn Metrization Theorem Proof Deconstruction
  Due 11/14/13 for grading: PS9.1

- PS10: H p.18 #2, PS10.1, PS10.2, H p.18 #3, PS10.3, H p.18 #9=M58.6, PS10.4, PS10.5
  - **PS10.1**: For each function f below, what familiar space is homeomorphic to space constructed from f? (In each case, prove your answer by pictures!)
    (a) Mapping cylinder for the function $f: I \to S^1$ defined by $f(t) = (\cos(4\pi t), \sin(4\pi t))$.
    (b) Mapping torus for the function $f: S^1 \to S^1$ defined by f(x,y) = (x,-y).
  - **PS10.2**: Let X_f be the mapping cylinder associated to a continuous function $f: X \to Y$. Let $j: Y \to X_f$ be the function $j(y) := [y]$ for all y in Y, and let $\overline{Y}$ be the image $j(Y)$.
    (a) Let $i: \overline{Y} \to X_f$ be the inclusion map and let $r: X_f \to \overline{Y}$ be defined by $r([(x,s)]) := [f(x)]$ and $r([y]) := [y]$ for all x in X, s in I, and y in Y. Show that r is a retraction from X_f to $\overline{Y}$, and that the set of functions $\{f_t : X_f \to X_f\}_{t \in I}$ defined by $f_t([(x,s)]) := [(x,t+(1-t)s)]$ and $f_t([y]) := [y]$, is a deformation retraction from X_f to $\overline{Y}$ - that is, a homotopy from the identity map on X_f to the composite function i o r, rel $\overline{Y}$. (Hint: You may use the results of Munkres Exercise 18.10 p. 112

and of Munkres Exercise 29.11 p. 186: If q:X -> Y is a quotient map and i':Z -> Z is the identity map for a compact Hausdorff space Z, then q x i': X x Z -> Y x Z is also a quotient map.)
(b) Show that *j* is an embedding, and that X_f is homotopy equivalent to *Y*.

- **PS10.3**: Define the paths f,g : I -> S^2 (with the Euclidean subspace topology) by f(s) := (cos(2 π s), sin(2 π s), 0) and g(s) := (1,0,0) for all s in I. Prove that f is homotopic to g rel {0,1}. (That is, show that f and g are "path homotopic".)
- **PS10.4**: Show that any contractible space is path-connected.
- **PS10.5**: Let X := {a,b} have the indiscrete topology. Compute the fundamental group $\pi_1(X,a)$. (Prove your answer.)

Due 11/26/13 for grading: PS10.1(b), H p.18 #9, PS10.5

- PS11: H p.38 #1, H p. 38 #3, PS11.1
  - **PS11.1**: (a) Show that for any continuous function h: (X,x_0) -> (Y,y_0), the induced map h_*: $\pi_1(X,x_0)$ -> $\pi_1(Y,y_0)$ is a well-defined group homomorphism.
  (b) Show that $(k \circ h)_* = k_* \circ h_*$
  (c) Show that $(1_{(X,x_0)})_* = 1_{\pi_1(X,x_0)}$.
  (d) Let f: X -> Y be a continuous map, let $x_0,x_1 \in X$, and let α be a path in X from x_0 to x_1. Let f_{*,x_0}: $\pi_1(X,x_0)$ -> $\pi_1(Y,f(x_0))$ and f_{*,x_1}: $\pi_1(X,x_1)$ -> $\pi_1(Y,f(x_1))$ be the maps induced by f at x_0 and x_1 respectively, and let β_α and $\beta_{f \circ \alpha}$ be the change of basepoint maps induced by the paths α in X and f ∘ α in Y, respectively. Prove that $\beta_{f \circ \alpha} \circ f_{*,x_0} = f_{*,x_1} \circ \beta_\alpha$. (This part is Hatcher's problem p. 39 # 15.)

Exam 2 12/5/13

S. Hermiller.