Spring 2003 prelim

1A. Let k be a field, and let $n \geq 1$. Prove that the following properties of an $n \times n$ matrix A with entries in k are equivalent:

(a) A is a scalar multiple of the identity matrix.

(b) Every nonzero vector $v \in k^n$ is an eigenvector of A.

2A. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, 0) = 0$ and

$$f(x,y) = \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2 + y^2}$$

for $y \neq 0$.

(a) Show that f is continuous at $(0, 0)$.

(b) Calculate all the directional derivatives of f at $(0, 0)$.

(c) Show that f is not differentiable at $(0, 0)$.

3A. Let $M_2(\mathbb{Q})$ denote the ring of 2×2 matrices with entries in $\mathbb{Q}$. Let R be the set of matrices in $M_2(\mathbb{Q})$ that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Prove that R is a subring of $M_2(\mathbb{Q})$.

(b) Prove that R is isomorphic to the ring $\mathbb{Q}[x]/(x^2)$.

4A. Prove that for each integer $n \geq 0$ there is a polynomial $T_n(x)$ with integer coefficients such that the identity

$$2\cos nz = T_n(2\cos z)$$

holds for all z.

5A. Let L be a real symmetric $n \times n$ matrix with 0 as a simple eigenvalue, and let $v \in \mathbb{R}^n$.

(a) Show that for sufficiently small positive real ϵ, the equation $Lx + \epsilon x = v$ has a unique solution $x = x(\epsilon) \in \mathbb{R}^n$.

(b) Evaluate $\lim_{\epsilon \to 0^+} \epsilon x(\epsilon)$ in terms of v, the eigenvectors of L, and the inner product $(\ ,\ )$ on $\mathbb{R}^n$.

6A. Let x_n be a sequence of real numbers so that $\lim_{n\to\infty}(2x_{n+1} - x_n) = x$. Show that $\lim_{n\to\infty} x_n = x$.

7A. (a) Suppose that H_1 and H_2 are subgroups of a group G such that $H_1 \cup H_2$ is a subgroup of G. Prove that either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

(b) Show that for each integer $n \geq 3$, there exists a group G with subgroups $H_1, H_2, \ldots, H_n$, such that no H_i is contained in any other, and such that $H_1 \cup H_2 \cup \cdots \cup H_n$ is a subgroup of G.

8A. Evaluate $\int_0^\infty e^{-x^2} \cos x^2 \, dx$.

9A. Let R be the set of complex numbers of the form

$$a + 3bi, \quad a, b \in \mathbb{Z}.$$

Prove that R is a subring of $\mathbb{C}$, and that R is an integral domain but not a unique factorization domain.

1B. (a) Prove that there is no continuously differentiable, measure-preserving bijective function $f\colon \mathbb{R} \to \mathbb{R}_{>0}$.

(b) Find an example of a continuously differentiable, measure-preserving bijective function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}_{>0}$.

2B. For an analytic function h on $\mathbb{C}$, let $h^{(i)}$ denote its i-th derivative. (If $i = 0$, then $h^{(i)} = h$.) Suppose that f and g are analytic functions on $\mathbb{C}$ satisfying

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f^{(0)} = 0$$
$$g^{(m)} + b_{m-1} g^{(m-1)} + \cdots + b_0 g = 0$$

for some constants $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in \mathbb{C}$. Show that the product function $F = fg$ satisfies

$$c_{mn} F^{(mn)} + c_{mn-1} F^{(mn-1)} + \cdots + c_0 F = 0$$

for some constants $c_0, \ldots, c_{mn} \in \mathbb{C}$ not all zero.

3B. Let f be an entire function such that $\operatorname{Re} f(z) \geq -2$ for all $z \in \mathbb{C}$. Show that f is constant.

4B. Suppose G is a nonabelian simple group, and A is its automorphism group. Show that A contains a normal subgroup isomorphic to G.

5B. Let C and D be nonempty closed subsets of $\mathbb{R}^n$, and assume that C is bounded. Prove that there exist points $x_0 \in C$ and $y_0 \in D$ such that $d(x_0, y_0) \leq d(x, y)$ for all $x \in C$, $y \in D$. Here $d(x, y)$ denotes the Euclidean metric on $\mathbb{R}^n$.

6B. Let $\mathrm{GL}_2(\mathbb{C})$ denote the group of invertible 2×2 matrices with coefficients in the field of complex numbers. Let $\mathrm{PGL}_2(\mathbb{C})$ denote the quotient of $\mathrm{GL}_2(\mathbb{C})$ by the normal subgroup $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$. Let n be a positive integer, and suppose that a, b are elements of $\mathrm{PGL}_2(\mathbb{C})$ of order exactly n. Prove that there exists $c \in \mathrm{PGL}_2(\mathbb{C})$ such that cac^{-1} is a power of b.

7B. Let $f(z)$ be a function that is analytic in the unit disk $D = \{|z| < 1\}$. Suppose that $|f(z)| \leq 1$ in D. Prove that if $f(z)$ has at least two fixed points z_1 and z_2 (that is, $f(z_j) = z_j$ for $j = 1, 2$), then $f(z) = z$ for all $z \in D$.

8B. Let $N = 30030$, which is the product of the first six primes. How many nonnegative integers x less than N have the property that N divides $x^3 - 1$?

9B. Let $A \subseteq \mathbb{R}$ be uncountable.

(a) Show that A has at least one accumulation point.

(b) Show that A has uncountably many accumulation points.

(Recall that a point is said to be an accumulation point of A if and only if it is the limit of a sequence of distinct terms from A.)