SPRING 2005 PRELIMINARY EXAMINATION SOLUTIONS

1A. (a) Let $(a_n)_1^\infty$ be a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Prove that $(a_n)_1^\infty$ is a Cauchy sequence.

(b) Is the converse true? Give a proof or a counterexample.

2A. Prove or disprove the statement: Every function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all x and y is continuous.

3A. Prove that there is no holomorphic bijection from the punctured disk $0 < |z| < 1$ in $\mathbb{C}$ onto the annulus $r < |z| < R$, where $0 < r < R < \infty$.

4A. Suppose A and B are commuting $n \times n$ matrices over $\mathbb{R}$. Suppose A and B are each diagonalizable over $\mathbb{R}$. Show that AB is diagonalizable over $\mathbb{R}$.

5A. Let I be an open interval and let $f \colon I \to \mathbb{R}$ have continuous k-th derivatives everywhere on I for all $k \leq n-1$. Let $a \in I$ be such that $f^{(k)}(a) = 0$ for $1 \leq k \leq n-1$, and assume that $f^{(n)}(a)$ is defined and $f^{(n)}(a) > 0$. Prove that if n is even, then f has a local minimum at a, and if n is odd, then f has no local extremum at a.

6A. For every positive integer n, define $[n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$. Prove that $[1]_q [2]_q \cdots [r]_q$ divides $[k+1]_q [k+2]_q \cdots [k+r]_q$ in the polynomial ring $\mathbb{Z}[q]$, for all positive integers k and r.

7A. Let U be a connected open subset of $\mathbb{C}$, and let $f(z)$ be a meromorphic function on U having at least one pole. For each $c \in U$ that is not a pole of f, let $R(c)$ be the radius of convergence of the Taylor series of f centered at c. Prove that $R(c)$ extends to a continuous function defined on all of U.

8A. Let C and D be two $n \times n$ positive definite Hermitian matrices over $\mathbb{C}$ and let $A = CD$. Prove that all eigenvalues of A are positive real numbers.

9A. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function that is zero outside some bounded subset of $\mathbb{R}^2$. Prove that

$$\lim_{\epsilon \to 0} \iint_{x^2+y^2 \geq \epsilon^2} \frac{f(x,y)}{(x+iy)^3} \, dx \, dy$$

exists.

1B. Let G be a finite group. Suppose $ab = ba$ holds whenever $a, b \in G$ have prime power order. Prove that G is abelian.

2B. Prove that, for any $\varepsilon > 0$, the function $f(z) = \sin z + \frac{1}{z+i}$ has infinitely many zeros in the strip $|\operatorname{Im} z| < \varepsilon$.

3B. Let $M_n(F)$ be the ring of $n \times n$ matrices over a field F. Prove that for every $A \in M_n(F)$ there exists $X \in M_n(F)$ such that $AXA = A$.

4B. Let D be a subset of $\mathbb{R}$, and let $f\colon D \to \mathbb{R}$ be a function. The graph of f is the subset

$$G := \{(x, y) : x \in D,\ y = f(x)\}$$

of $\mathbb{R}^2$. Prove that if G is compact, then f is continuous.

5B. Let $\mathbb{Q}(x)$ be the field of rational functions in one variable over $\mathbb{Q}$. Let $i\colon \mathbb{Q}(x) \to \mathbb{Q}(x)$ be the unique field automorphism such that $i(x) = x^{-1}$. Prove that the fixed subfield $\{r \in \mathbb{Q}(x) : i(r) = r\}$ is equal to $\mathbb{Q}(x + x^{-1})$.

6B. Evaluate the integral $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$, where $a > 0$.

7B. Let $\mathbb{F}_p$ be the field of p elements. Let $\mathrm{SL}_2(\mathbb{F}_p)$ be the group of 2×2 matrices over $\mathbb{F}_p$ of determinant 1. Let G be a normal subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$. Suppose G contains a non-identity element γ that fixes a nonzero vector v. Show that any $\gamma' \in \mathrm{SL}_2(\mathbb{F}_p)$ that fixes a nonzero vector v' belongs to G.

8B. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Suppose that $f(0) = 0$, and that $|f'(x)| \le |f(x)|$ for all $x \in \mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

9B. (a) Prove that if $n > 0$ is even, there does not exist $f(x) \in \mathbb{R}[x]$ such that $f(x)^2 - x$ is divisible by $x^n - 1$.

(b) For odd $n > 0$, find the number of $f(x) \in \mathbb{R}[x]$ of degree $< n$ such that $f(x)^2 - x$ is divisible by $x^n - 1$.