Final Exam Practice 1, May 8, 2018 **Math 21b, Spring 2018**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F All symmetric matrices with positive entries are diagonalizable.

Solution:
All symmetric matrices are diagonalizable, whether they have positive entries or not.

2) T F If A is a symmetric matrix, then A is invertible.

Solution:
Take a 2×2 matrix which is a projection onto a line. It is symmetric but not invertible.

3) T F If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

Solution:
It would not be true for non-symmetric matrices but symmetric matrices are diagonalizable and A^5 becomes the diagonal B^5 in the eigenbasis. Now $B^5 = 0$ implies $B = 0$.

4) T F If A and B are $n \times n$ symmetric matrices, then $A + B$ is symmetric.

Solution:
By definition $(A+B)_{ij} = A_{ij} + B_{ij}$ so that $(A+B)^T = A^T + B^T$.

5) T F If A and B are $n \times n$ symmetric matrices, then AB is symmetric.

Solution:
Take for A a diagonal 2×2 matrix with different diagonal entries and for B a rotation dilation with $a = 1, b = 1$. Then $AB \neq BA$.

6) T F If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a stable equilibrium.

Solution:
The negative determinant actually implies that the signs of the eigenvalues differ. Take A the reflection at the x axes for example. Now there are eigenvalues $-1, 1$ and the equilibrium is not stable for the continuous dynamical system.

7) T F If A is any matrix, then both AA^T and A^TA are diagonalizable.

Solution:
Yes, they are both symmetric.

8) T F The matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

Solution:
The algebraic multiplicity of the eigenvalue 2 is 2, the geometric multiplicity is 1.

9) T F Every 2×3 matrix has a non-zero vector in its kernel.

Solution:
There can only be 2 leading one

10) T F If the image of A is contained in the kernel of a square matrix A, then $A^2 = 0$.

Solution:
Yes, $A^2x = A(Ax) = 0$ as Ax is in the image and so part of the kernel.

11) T F If a 2×2 matrix A is diagonalizable $A^7 + A + I_2$ is diagonalizable.

Solution:
One can use the same S matrix.

12) T F The function $f(x) = 7\sin(5x)$ has the Fourier coefficient $b_7 = 5$ if $f(x) = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + b_n \sin(nx)$.

Solution:
It is the coefficient $b_5 = 7$.

13) T **F** The space of smooth functions f satisfying $f(x) = x + \sin(f(x))$ forms a linear space.

Solution:
It does not contain $f = 0$.

14) **T** F If the geometric multiplicity of the eigenvalue 1 is 2 for a 2×2 matrix A then A is the identity matrix.

Solution:
The geometric multiplicity is the nullity $A - I_2$. A 2×2 matrix of nullity 2 is the zero matrix. As $A - I_2$ is the zero matrix, A is the identity matrix.

15) T **F** Let A, B be arbitrary 2×2 matrices. Then some eigenvalue of AB is the product of some eigenvalues of A and B.

Solution:
There is no relation in general. Just make up an example, it most likely fails. like $\{\begin{bmatrix} 7 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\}$.

16) **T** F If a continuous dynamical system $x'(t) = Ax(t)$ with a 2×2 symmetric matrix A is asymptotically stable then $\det(A) > 0$.

Solution:
Both eigenvalues are negative then.

17) **T** F If $\vec{x}(t+1) = A\vec{x}(t)$ is an asymptotically stable dynamical system, then each eigenvalue λ of A satisfies $|\lambda| < 1$.

Solution:
Yes, we know that from the closed form solution.

18) **T** F The function $f(x,t) = \sin(5x)e^{-25t}$ solves the heat equation $f_t = f_{xx}$.

Solution:
It indeed does

19) T F The length $||6\sin(175x) + 8\cos(199x)||$ is equal to 10.

Solution:
This is a consequence of the Parseval equality.

20) T F All solutions to the differential equation $x''(t) + x(t) = \sin(t)$ stay bounded for $t \to \infty$.

Solution:
This is a resonance case. The solution grows linearly.

Problem 2) (10 points) No justifications needed

a) (4 points) Which properties do hold for the following matrix? (Remember the circular matrices?)

$$A = \begin{bmatrix} 100 & 2 & 3 & 4 \\ 4 & 100 & 2 & 3 \\ 3 & 4 & 100 & 2 \\ 2 & 3 & 4 & 100 \end{bmatrix}$$

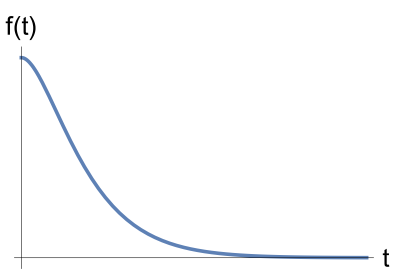
Property	Check if applies
The matrix A is symmetric.	
The matrix A has eigenvector $[1, i, i^2, i^3]^T$.	
The matrix A has an eigenvalue $100 + 2i - 3 - 4i$.	
The matrix A is invertible.	
The matrix A is diagonalizable over the complex numbers.	
The matrix A is diagonalizable over the real numbers.	

Solution:
It is not symmetric and not diagonalizable over the reals. The rest is ok.

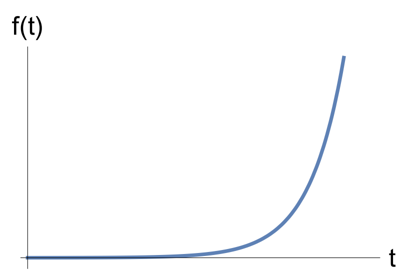
b) (3 points) Match the differential equations with possible solution graphs. There is an exact match.

Enter A-D	Differential equation
	$f'(t) + f(t) = e^{-t}$
	$f'(t) - f(t) = e^{-t}$

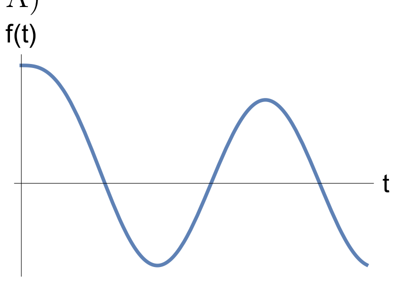
Enter A-D	Differential equation
	$f''(t) + f(t) = e^{-t}$
	$f''(t) = e^{-t}$



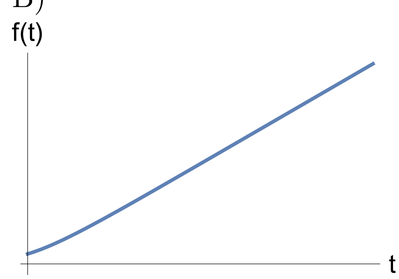
A)



B)



C)



D)

Solution:

Enter A-D	Differential equation
A	$f'(t) + f(t) = e^{-t}$
B	$f'(t) - f(t) = e^{-t}$

Enter A-D	Differential equation
C	$f''(t) + f(t) = e^{-t}$
D	$f''(t) = e^{-t}$

c) (3 points) Pick the statements which are true for a 3×3 matrix A.

Statement	Check if true
If A is diagonalizable and A has real eigenvalues then A is symmetric	
If A is symmetric, then A is diagonalizable and A has real eigenvalues	
If A has only eigenvalues of algebraic multiplicity 1, then A is diagonalizable	
If A is diagonalizable, then all eigenvalues of A have algebraic multiplicity 1	

Solution:
Only the second and third statements are true.

Problem 3) (10 points) No justifications needed

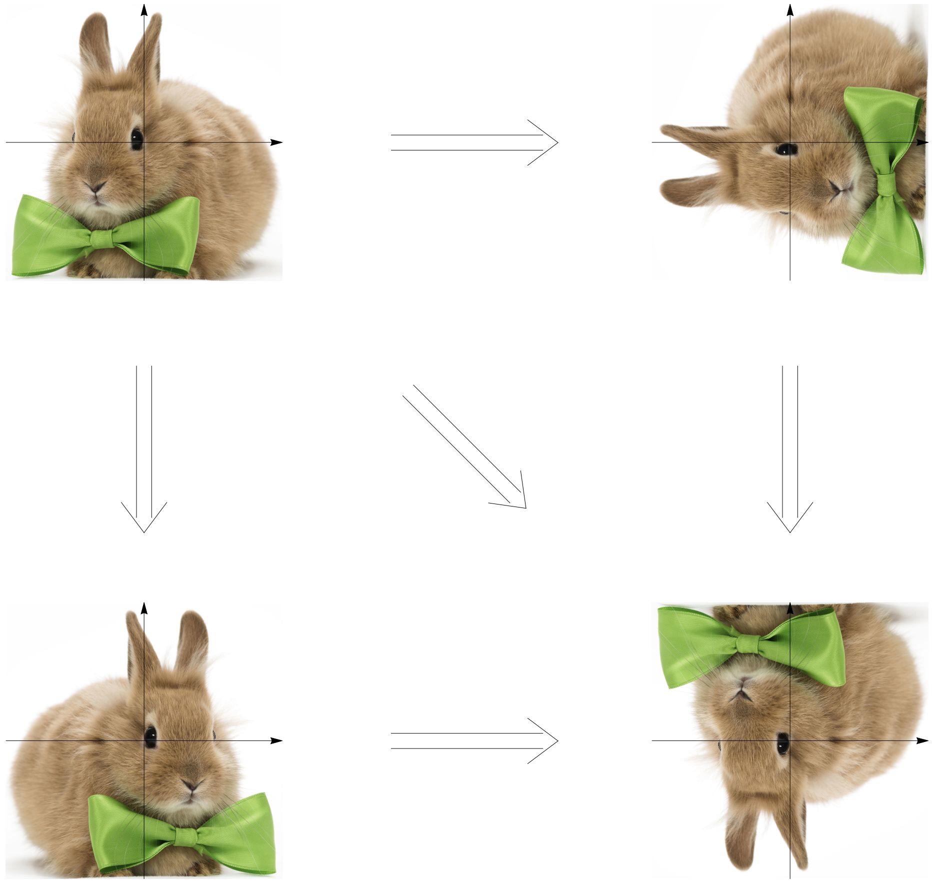
a) (2 points) Linear or not linear?

Space	linear	nonlinear
$\{f \in C^\infty \mid f(1) = 1\}$		
$\{f \in C^\infty \mid f'(1) = 0\}$		

Transformation	linear	nonlinear
$Tf(x) = f'(x)\sin(x)$		
$Tf(x) = f'(3)f(x)$		

Solution:
Left: the first is not linear. The second is linear.
Right. the first is linear. The second is not linear.

b) (5 points) Match each of the 5 arrows with a matrix that does the indicated transformation. Place the name of the matrix $A - F$ near each arrow. Of course, one of the matrices is not used.



$$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\ B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ C=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\ D=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\ E=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ F=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Solution:

*	A	*
C	E	B
*	D	*

c) (3 points) Label each function with the most accurate description of its Fourier series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\ .$$

Function	Check if $a_0 = 0$	Check if all $a_n = 0$	Check if all $b_n = 0$
$f(x) = \sin^3(x)$			
$f(x) = \sin^2(x)$			
$f(x) = \sin^2(x) + \sin^3(x)$			

Solution:

Function	Check if $a_0 = 0$	Check if all $a_n = 0$	Check if all $b_n = 0$
$f(x) = \sin^3(x)$	X	X	
$f(x) = \sin^2(x)$			X
$f(x) = \sin^2(x) + \sin^3(x)$			

Problem 4) (10 points)

a) (8 points) Solve the following linear system $A\vec{x} = \vec{b}$ of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

by row reducing the augmented matrix. We want you to perform the actual row reduction steps.

Solution:
a) The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Lets first subtract the first row from all the others:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Now subtract the second row from the 4th and 6th

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Now subtract the third from the fourth and the 5th from the sixth

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Now we can see the solution to the right.

b) (2 points) Change one single entry in the first row of the matrix A leaving the vector $\vec{b}$ alone to get a matrix B so that the modified system $B\vec{x} = \vec{b}$ has no solution.

The **binary Pascal matrix** A was constructed row by row, recursively adding up all the entries up left or up but taking remainders when dividing by 2.

The picture shows **Blaise Pascal** (1623 - 1662), who in 1653 produced a tabular representation of Binomial coefficients called today the **Pascal triangle**.

Solution:
b) Just make the first row of A zero:

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

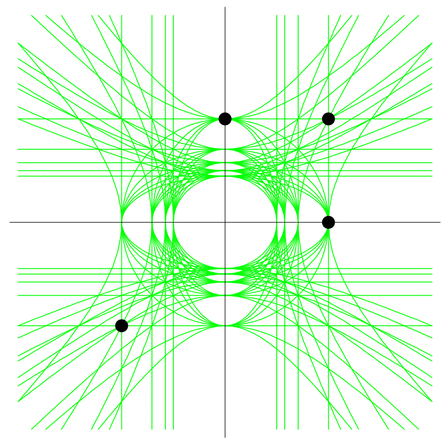
Problem 5) (10 points)

Using the least square method, find the function

$$ax^2 + by^2 = 1$$

which best fits the data points in the following table

x	y
1	0
0	1
1	1
-1	-1



Solution:
Write down the systems of equations (don't leave out any of the equations. $a = 1, b = 1, a + b = 1, a + b = 1$. It is a major mistake to discard the last equation.) The matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The vector $\vec{b} = [1, 1, 1, 1]^T$ works. Now kick in the formula. First compute $A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^T \vec{b} = [3, 3]^T$. The result is $a = 3/5, b = 3/5$. This is not a surprise as the data were symmetric.

Problem 6) (10 points)

You have an **investment account** x and a **credit account** y and at time $t = 0$ the account data: $x(0) = 2000$, $y(0) = 0$. The time dynamics of the portfolio is described by the following matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

We look at two models, the case of discrete time and the case of continuous time.

a) (4 points) Find a closed form solution to the problem

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with the initial condition given above.

b) (4 points) Find a closed form solution to the problem

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with the same given initial condition.

c) (2 points) In which of the two portfolios does $x(t)$ grow faster? Is it a) or b)? No explanations are needed in c).

Solution:
a) The eigenvalues of A are 3 and 1 (the sum of the row entries are 3 the trace is 4). The eigenvectors are $[1, 1]^T$ and (symmetric matrix) the orthogonal $[1, -1]^T$. Now write

$$\begin{bmatrix} 2000 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which gives $c_1 = 1000$ and $c_2 = -1000$. Now we can write down the closed form solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 1000 \cdot 3^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1000 \cdot 1^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b) We can start with the same equation (1) and get

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 1000e^{3 \cdot t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1000e^{1 \cdot t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

c) Now e^{3t} grows faster than $3^t = e^{t \log(3)}$.
It is a general principle that continuous compound interest rates grow faster than discrete compound interest, where your portfolio gets updated monthly or yearly.

Problem 7) (10 points)

The following magic square is called **Chautisa Yantra**:

$$A = \begin{bmatrix} 7 & 12 & 1 & 14 \\ 2 & 13 & 8 & 11 \\ 16 & 3 & 10 & 5 \\ 9 & 6 & 15 & 4 \end{bmatrix}.$$

It has been inscribed into a stone during the 10th century and can now be found in the Parshvanath temple in Khajuraho, India.

a) (3 points) The vector $\begin{bmatrix} -1 \\ -3 \\ 1 \\ 3 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

b) (3 points) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is its corresponding eigenvalue?

c) (2 points) What is the sum of all the four eigenvalues $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ of A?

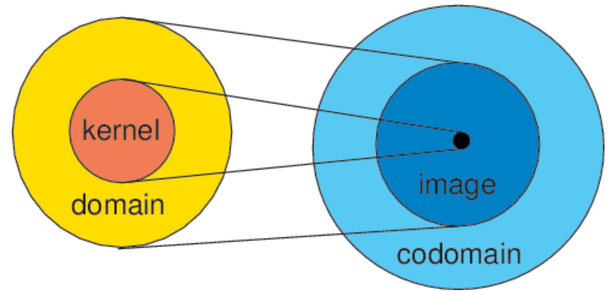
d) (2 points) What is the determinant of A?

Solution:
a) Just multiply Av. We get 0. The eigenvalue is 0.
b) The eigenvalue is 34, the sum of the row entries.
c) The sum of the eigenvalues is 34, the sum of the diagonal entries.
d) The determinant is the product of the eigenvalues which is 0.

Problem 8) (10 points)

a) (4 points) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} .$$



b) (3 points) Find a basis for the kernel of A.

c) (3 points) Find a basis for the image of A.

Solution:
a) The eigenvalues are $\{0, 0, 0, 10\}$. (Row reduction showed that there were 3 eigenvalues 0. The sum of the eigenvalues is 10). The eigenvectors are $[-2, 1, 0, 0]^T, [-3, 0, 1, 0]^T, [-4, 0, 0, 1]^T, [1, 1, 1, 1]^T$.
b) $\mathcal{B} = \{[-2, 1, 0, 0]^T, [-3, 0, 1, 0]^T, [-4, 0, 0, 1]^T\}$.
c) $\mathcal{B} = \{[1, 1, 1, 1]^T\}$, they span the image.

Problem 9) (10 points)

To celebrate the launch of our new determinant **bumper sticker**, we solve some determinants!
As always, we need not only your answer to the problem but also the path which led to the solution.

a) (2 points) Find the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} .$$

Solution:
Row reduce. The determinant is -1.

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 9 & 2 & 2 & 2 \\ 2 & 9 & 2 & 2 \\ 2 & 2 & 9 & 2 \\ 2 & 2 & 2 & 9 \end{bmatrix}.$$

Solution:
The eigenvalues of $A - 7I$ are $0, 0, 0, 8$. The eigenvalues of A are $7, 7, 7, 15$. The determinant is $7^3 \cdot 15 = 5145$.

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 2 & 8 & 5 & 2 \\ 2 & 9 & 2 & 2 \\ 2 & 8 & 9 & 2 \\ 2 & 2 & 1 & 2 \end{bmatrix}.$$

Solution:
Two columns are the same. The determinant is $\boxed{0}$.

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 2 & 5 & 4 \\ 2 & 5 & 4 & 3 \end{bmatrix}.$$

Solution:
There is only one pattern leading to 2^4 with 6 upcrossings. The answer is 16.

e) (2 points) Find the determinant of

$$E = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 3 \\ 4 & 1 & 4 & 1 \end{bmatrix}.$$

Solution:
This is a partitioned matrix. The determinant is $(2-12)(2-12) = 100$.

Problem 10) (10 points)

You are a chef in the first season of the new reality TV series "21b's kitchen". Solve the following differential equations:

a) (2 points)

$$f''(t) = t$$

b) (3 points)

$$f''(t) + f(t) = t$$

c) (3 points)

$$f''(t) - f(t) = t$$

d) (2 points) Check for each of the following two equations whether it has only solutions which remain bounded for $t \to \infty$? We need no explanation in d).

Differential equation	has only bounded solutions for $t \to +\infty$
$f''(t) + f'(t) = \cos(t)$	
$f''(t) - f'(t) = \cos(t)$	

Solution:
All can be done by cookbook. The first part a) also by integration.
a) Just integrate twice $C_1 + C_2t + t^3/6$
b) This is a harmonic oscillator $C_1 \cos(t) + C_2 \sin(t) + t$.
c) $C_1e^t + C_2e^{-1} - t$. d) In the first case we have $C_1 + C_2e^{-t}$ as a homogenous solution. The special solution is $\sin(t)/2 - \cos(t)/2$. In the second case we have $C_1t + C_2e^t$ as a homogeneous solution and $-\cos(t)/2 - \sin(t)/2$ as a special solution. Due to the e^t there are unbounded solutions. The first one has only bounded solutions for $t \to +\infty$.

Problem 11) (10 points)

We consider the nonlinear system of differential equations

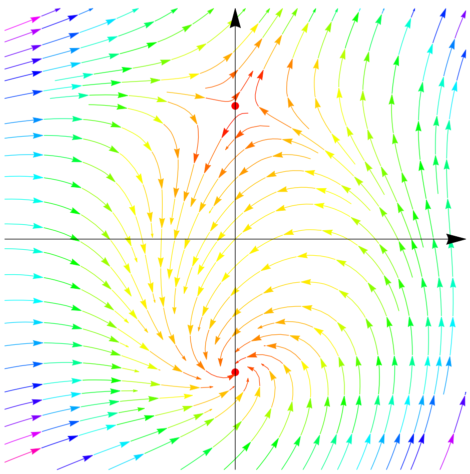
$$\begin{aligned} \frac{d}{dt}x &= (x-1)^2 + y^2 - 10 \\ \frac{d}{dt}y &= (x+1)^2 + y^2 - 10 \,. \end{aligned}$$

a) (2 points) Find the equilibrium points.

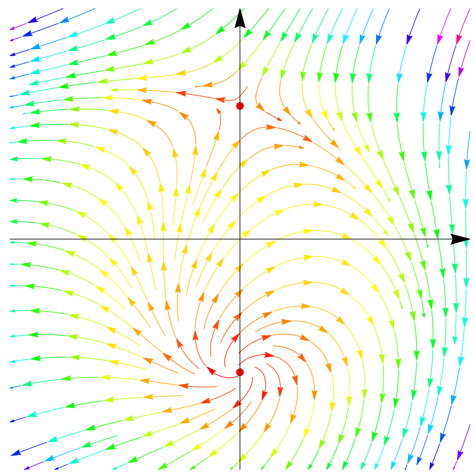
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Analyze the stability at each equilibrium point.

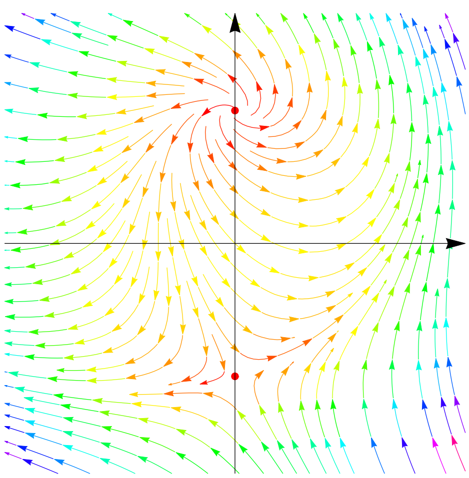
d) (2 points) Which of the following four phase portraits A-D matches?



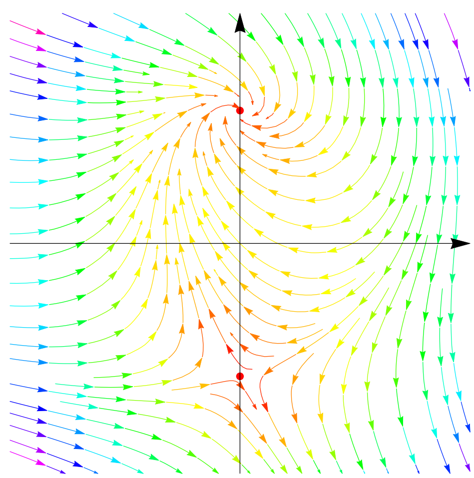
A



B



C



D

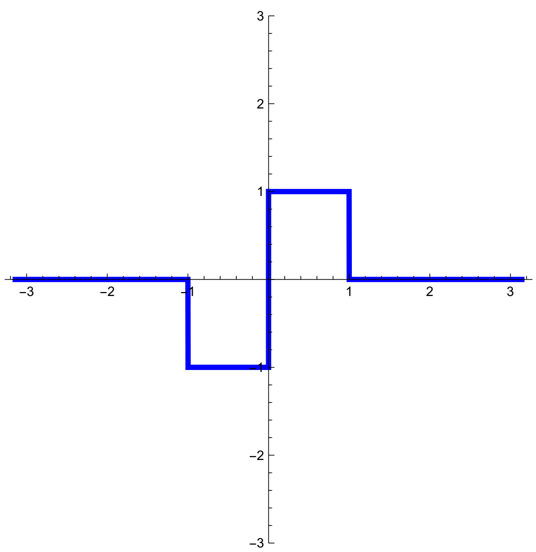
Solution:
a) The intersection of the nullclines (circles) are $(0,3)$ and $(0,-3)$.
b) The Jacobian matrix is $J(x,y) = \begin{bmatrix} 2(x-1) & 2y \\ 2(x+1) & 2y \end{bmatrix}$. c) At the point $(0,3)$, this is $\begin{bmatrix} -2 & 6 \\ 2 & 6 \end{bmatrix}$ and there is a positive and negative eigenvalue ($det < 0$).
At the point $(0,-3)$, this is $\begin{bmatrix} -2 & -6 \\ 2 & -6 \end{bmatrix}$ which has negative real part and some imaginary part. This spirals in.
d) The first phase portrait matches.

Problem 12) (10 points)

a) (6 points) Find the **Fourier series** of the function

$$f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ -1 & -1 \leq x \leq 0 \\ 0 & |x| > 1 \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Solution:
a) The function is odd. It has a sin-series. We have

$$b_n = \frac{2}{\pi} \int_0^1 \sin(nx)\, dx .$$

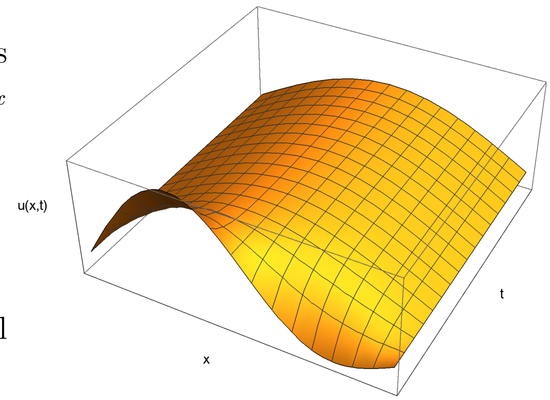
This is equal to $\boxed{2[1-\cos(n)]/(n\pi)}$.
b) By Parseval, we know the sum of the squares $\sum_{n=1}^{\infty} b_n^2$ equal to $(2/\pi) \int_0^1 1^2\, dx = \boxed{2/\pi}$

Problem 13) (10 points)

a) (5 points) Solve the **modified heat equation**

$$u_t = u_{xx} - u_{xxxx} + t$$

with initial condition $u(x,0) = 4\sin(x) + 5\sin(2x)$. As usual, first find a solution of the PDE $u_t = u_{xx} - u_{xxxx}$ then add a particular solution.



b) (5 points) Solve the **modified wave equation**

$$u_{tt} = u_{xx} - u_{xxxx} + t$$

with initial condition $u(x,0) = 7\sin(5x)$ and the initial velocity

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx) \ .$$

Solution:
a) This is a routine problem as we have done many. Ignore first the t part. The operator $D^2 - D^4$ used on the right hand side has eigenvectors $\sin(nx)$ with eigenvalues $-n^2 - n^4$. We can write down the solution with initial position:

$$u(x,t) = 4\sin(x)e^{(-1^2-1^4)t} + 5\sin(2x)e^{(-2^2-2^4)t}$$

Now add a particular solution (integrate $u_t = t$ to get $u = t^2/2$. Note that the constant C_1 is zero as we don't want to spoil the initial condition. The full solution is

$$u(x,t) = 4\sin(x)e^{(-1^2-1^4)t} + 5\sin(2x)e^{(-2^2-2^4)t} + t^2/2$$

b) First write down the solution with initial position,

$$u_{position}(x,t) = 7\sin(5x)\cos(\sqrt{(5^2+5^4)}t)$$

Now write down the solution with initial velocity

$$u_{velocity}(x,t) = \sum_n \frac{1}{n^2}\sin(nx)\frac{\sin(\sqrt{n^2+n^4}t)}{\sqrt{n^2+n^4}}$$

Finally find a particular solution, taking care of the driving force t:

$$u_{particular}(t) = t^3/6 \ .$$

The solution is the sum of all these

$$u(x,t) = u_{position}(x,t) + u_{velocity}(x,t) + u_{particular}(t) \ .$$

Note that the constants $C_1 t + C_2$ are both zero as we don't want to spoil initial position and velocity.

Problem 14) (10 points)

Find the determinant of the following 124×124 matrix. This is the largest matrix which has ever appeared in a practice exam. It has 15376 entries. We like it extreme.

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix}$$

Solution:
The matrix is a diagonal matrix. The determinant is 2^{124}. We just wanted to go for the record.