Name:

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MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ is symmetric.

Solution:
Direct check.

2) T F The vector $\vec{v} = 0$ is always in the kernel of a matrix A.

Solution:
By definition, $v = 0$ is never an eigenvector

3) T F For any 2×2 matrices A, the kernel of A is a subspace of the image of A.

Solution:
This is already false for the zero matrix

4) T F If a 2×2 matrix A is diagonalizable then the matrix A^T is diagonalizable.

Solution:
$A = S^{-1}BS$ implies $A^T = (S^T)^{-1}B^TS^T$.

5) T F The Fourier series of the function $f(x) = \sin^2(x)$ only involves cosine terms and the constant term.

Solution:
The function is even.

6) T F The space of smooth functions satisfying the equation $f(x) = f(-x)^2$ forms a linear space.

Solution:
The function $f = 1$ satisfies the equation but $f = 2$ not.

7) T **F** The algebraic multiplicity of an eigenvalue 0 of A is equal to the nullity of A.

Solution:
The geometric multiplicity is the nullity.

8) T **F** The sum $A+B$ of two real matrices A, B which both have eigenvalues $\lambda = i, -i$ is a matrix with eigenvalues $i, -i$.

Solution:
Take a matrix A and then $-A$.

9) T **F** A discrete dynamical system $x(t+1) = Ax(t)$ with a 2×2 matrix A is asymptotically stable if $\det(A) < 0$.

Solution:
We need $|\det(A)| < 1$, as for discrete dynamical systems, the eigenvalues should be smaller than 1 in norm.

10) **T** F If $x'(t) = Ax(t)$ and $x(t+1) = Ax(t)$ are both asymptotically stable, then all real eigenvalues λ satisfy $-1 < \lambda < 0$.

Solution:
Yes, this assures it

11) T **F** The sum of two orthogonal projections is an orthogonal projection.

Solution:
Take a projection P and form $P+P$. This is no more a projection.

12) T **F** The function $f(x,t) = e^{-6t}\sin(3x)$ solves the heat equation $f_t = 2f_{xx}$.

Solution:
Yes, just check this

13) **T** F If a system of linear equations $A\vec{x} = \vec{c}$ with 2×2 matrix A has infinitely many solutions, then there exists $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution.

Solution:
There is a nontrivial kernel. The image is therefore not the entire space.

14) | T | **F** | The equilibrium point $(0, 0)$ of the nonlinear system $x' = x^2$, $y' = y^2$ is asymptotically stable.

Solution:
It is not stable because the eigenvalues are zero.

15) | **T** | F | All real symmetric matrices are diagonalizable over the real numbers.

Solution:
This is the spectral theorem

16) | T | **F** | $||3\sin(5x) - 7\sin(10x)|| = \sqrt{5^2 + 10^2}$, where the length $||f||$ of f is defined by the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx$.

Solution:
This is Parseval's equality. But the result is $\sqrt{3^2 + 7^2}$

17) | T | **F** | The sum of the eigenvalues of a symmetric 2×2 matrix is 0.

Solution:
Take a matrix with eigenvalues $0, 1$.

18) | **T** | F | The differential equation $x''(t) + 9x(t) = \sin(3t)$ with $x(0) = 2, x'(0) = 2$ has solutions for which $|x(t)|$ becomes arbitrary large.

Solution:
This is a resonance case.

19) | T | **F** | For any invertible matrix, A^{-1} is similar to A.

Solution:
The eigenvalues are $1/\lambda$.

20) T **F** If an $n \times n$ matrix has the property that the sum of all matrix entries are zero, then the matrix is not invertible.

Solution:
A counter example is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Problem 2) (10 points) No justifications needed

a) (6 points) We are given a real 3×3 matrix A and define $B = A^3 + A$.

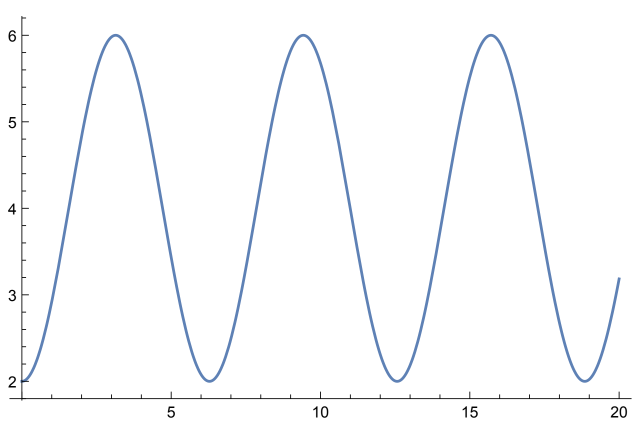
Statement	Always true
If A is invertible, then B is invertible	
If A is diagonalizable, then B is diagonalizable	
If A is symmetric, then B is symmetric	
If A has a zero determinant, then B has a zero determinant	
If A has zero traces, then B has zero trace	
If A has an eigenvalue 1, then B has an eigenvalue 2	

Solution:

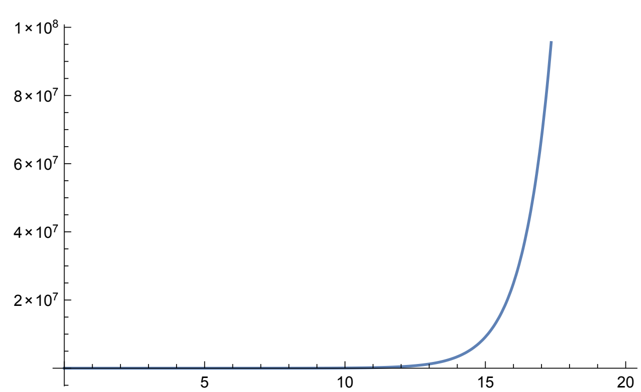
Statement	Always true
If A is invertible, then B is invertible	
If A is diagonalizable, then B is diagonalizable	*
If A is symmetric, then B is symmetric	*
If A has a zero determinant, then B has a zero determinant	*
If A has zero traces, then B has zero trace	
If A has an eigenvalue 1, then B has an eigenvalue 2	*

b) (4 points) Match the differential equations with possible solution graphs.

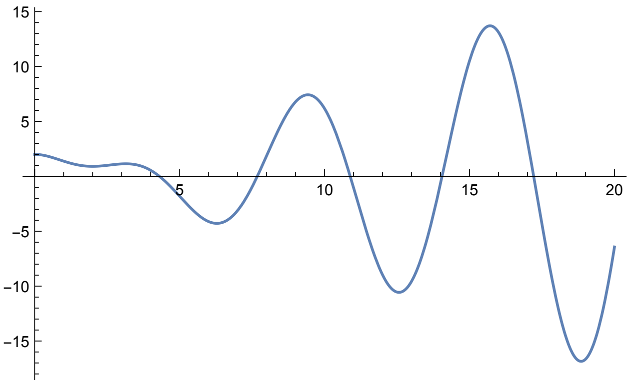
Enter A-D	Differential equation
	$f''(t) + f(t) = 2\sin(2t)$
	$f'(t) = 2\sin(t)$
	$f'(t) - f(t) = 2\sin(2t)$
	$f''(t) + f(t) = 2\sin(t)$



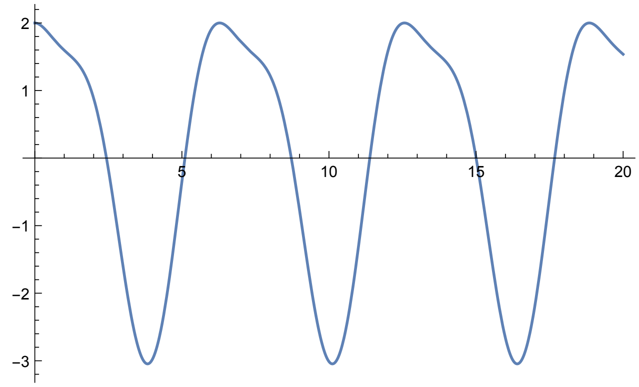
A)



B)



C)



D)

Solution:
DABC

Problem 3) (10 points) No justifications needed

a) (6 points) Assume T is a transformation on C^∞, the linear space of smooth functions on the real line. Which of the following transformations are linear?

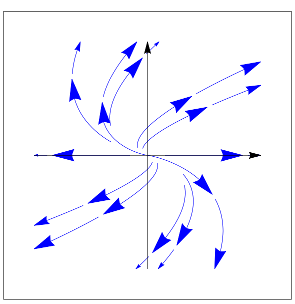
Transformation	Check if linear
$Tf(x) = 3f(5x^2)$	
$Tf(x) = f(4-x)$	
$Tf(x) = \sin(x)f'(x)$	
$Tf(x) = f(5)f(x)$	
$Tf(x) = f(5)x^2$	
$Tf(x) = 1 + f(5f(x))$	

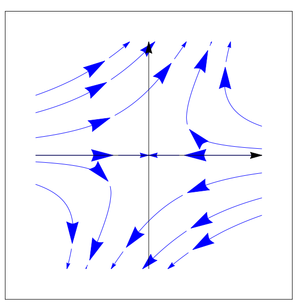
Solution:

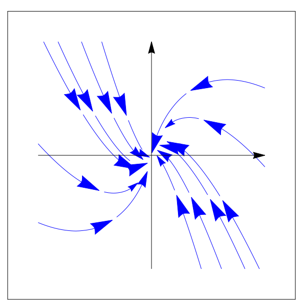
Transformation	Check if linear
$Tf(x) = 3f(5x^2)$	X
$Tf(x) = f(4-x)$	X
$Tf(x) = \sin(x)f'(x)$	X
$Tf(x) = f(5)f(x)$	
$Tf(x) = f(5)x^2$	X
$Tf(x) = 1 + f(5f(x))$	

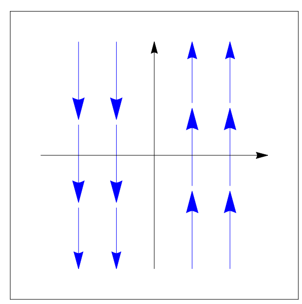
b) (4 points) Match the differential equation $\frac{d}{dt}x(t) = Ax(t)$ with the phase portraits. There is an exact match.

Matrix	Enter a) - d)
$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$	
$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	


a)


b)


c)


d)

Solution:
c)b)d)a)

Problem 4) (10 points)

a) (4 points) Find all the solutions to the following system of linear equations:

$$\left| \begin{array}{ccccccccccccc} x & + & y & + & z & + & u & + & v & + & w & = 6 \\ x & - & y & + & z & - & u & + & v & - & w & = 0 \\ x & + & y & + & z & + & u & - & v & - & w & = 2 \end{array} \right|$$

b) (3 points) The system in a) can be written in matrix form as $A\vec{x} = \vec{b}$. Find a basis for the kernel of A.

c) (3 points) Find a basis for the image of A.

Solution:
a) Row reduce the augmented matrix to get the pivot columns and free variables:

$$\begin{array}{ccccccc} & & r & s & & t & \\ \hline \end{array}$$
$$\left[\begin{array}{ccccccc} \boxed{1} & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 2 \end{array} \right].$$

Write this again as a system using the free variables to get The solution is

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

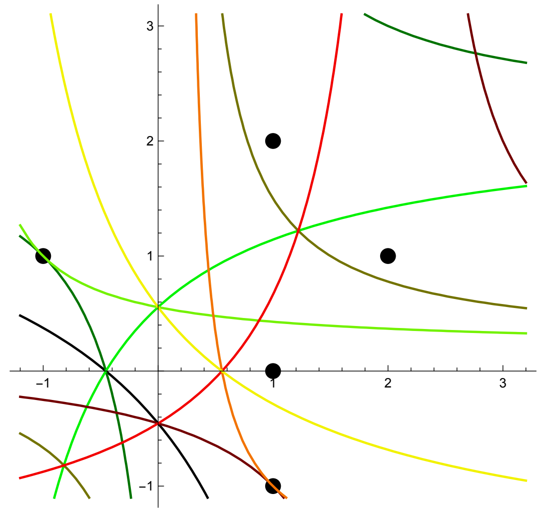
b) A basis for the kernel is given by the vectors which come with the free variables:

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

c) A basis for the image is given by the pivot columns

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Problem 5) (10 points)



Using the least square method, find the hyperbola

$$xy + ax + by = 1$$

which best fits the data points (x, y):

$$\{(1,0), (-1,1), (1,2), (1,-1), (2,1)\ \} \ .$$

Solution:
write down the equations:

$$\begin{aligned} a &= 1 \\ -a + b &= 2 \\ a + 2b &= -1 \\ a - b &= 2 \\ 2a + b &= -1 \end{aligned}$$

This leads to the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 2 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$$

and the vector $b = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}$. We get the solution with $x = (A^T A)^{-1} A^T b$. We have $A^T A)^{-1} = \begin{bmatrix} 7 & -2 \\ -2 & 8 \end{bmatrix}/52$ and $A^T b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$. Now $x = \begin{bmatrix} -2/13 \\ -5/13 \end{bmatrix}$. The best hyperbola is $xy - (2/13)x - (5/13)y = 1$.

Problem 6) (10 points)

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

a) (3 points) Find a matrix B which is diagonal and similar to A.

b) (3 points) Find a matrix S such that $B = S^{-1}AS$ is that diagonal matrix obtained in a).

c) (2 points) Solve the discrete dynamical system $\vec{v}(t+1) = A\vec{v}(t)$, for which the initial condition is $\vec{v}(0) = \begin{bmatrix} 11 \\ 11 \\ 2 \end{bmatrix}$.

d) (2 points) Is the system defined in c) stable?

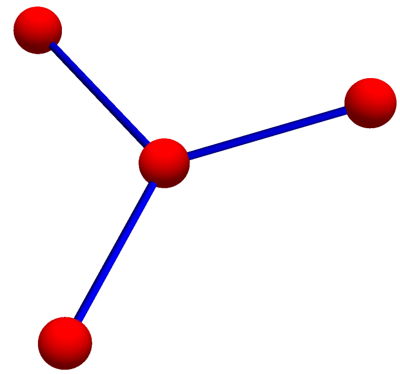
Solution:
a) Since the eigenvalues are $\boxed{5, -1, 3}$, we have the similar diagonal matrix $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
b) S contains the eigenvectors of A

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} .$$

c) Since $\begin{bmatrix} 11 \\ 11 \\ 2 \end{bmatrix} = 12 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, we have $v(t) = 12 \cdot 5^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3^t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. The system is $\boxed{\text{not stable}}$.

Problem 7) (10 points)

The following matrix is called the **Laplacian of the star graph"**



$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} .$$

a) (2 points) You are told that $\mathcal{B} = \{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$ is an eigenbasis of A.

Find the eigenvalues of A.

b) (2 points) Find the characteristic polynomial of A.

c) (3 points) Write down the solution to $\vec{v}\,'(t) = A\vec{v}(t)$ with initial condition $\vec{v}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Side remark: You just have solved the "heat equation" on the star graph.

d) (3 points) Write down the solution to $\vec{v}(t+1) = A\vec{v}(t)$ with initial condition $\vec{v}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution:
a) The eigenvalues are $-4, -1, -1, 0$.
b) The characteristic polynomial is $(-4-\lambda)(-1-\lambda)(-1-\lambda)(0-\lambda)$.
c) Since $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -3/4 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (3/4) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we have

$$\vec{v}(t) = (-3/4)e^{-4t} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (3/4)e^{-0t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

d)

$$\vec{v}(t) = (-3/4)(-4)^t \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (3/4)0^t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

Problem 8) (10 points)

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$.

a) (2 points) Compute A^2 and A^{-1}.

b) (2 points) Find the eigenvalues of $A^T + A$.

c) (2 points) Find the QR decomposition of A.

d) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

e) (2 points) What are the algebraic and geometric multiplicities of the eigenvalues of A? Is A diagonalizable?

Solution:
a) $A^2 = \begin{bmatrix} 4 & 8 \\ 0 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1/2 \end{bmatrix}$.
b) $A + A^T = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. $\lambda = 6, 2$.
c) The QR decomposition is $A = I_2 A$.
d) The characteristic polynomial is $\lambda^2 - 4\lambda + 4$.
e) The algebraic multiplicity of 2 is 2, the geometric multiplicity 1. The matrix is not diagonalizable.

Problem 9) (10 points)

Remember the song:

"Laplace, Row Reduce" *Partitions, Triangular* *Eigenvectors, Eigenvalues* *Patterns you can use!"*

a) (2 points) Find the determinant of the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

b) (2 points) Find the determinant of the following GCD matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{bmatrix} .$$

c) (2 points) Find the determinant of the following matrix

$$\begin{bmatrix} 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 4 & 4 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

d) (2 points) Find the determinant of the following matrix

$$\begin{bmatrix} 6 & 1 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix} .$$

e) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:
Follow the song!
a) 80 b) 4 c) 2 d) $10 \cdot 5^4 = 6250$ e) 2^9.

Problem 10) (10 points)

Find the general solution to the following differential equations:

a) (2 points)

$$f'(t) + f(t) = t^2 + 1$$

b) (2 points)

$$f''(t) + 9f(t) = t^2 + 1$$

c) (3 points)

$$f''(t) + 2f'(t) + f(t) = t^2$$

d) (3 points)

$$f''(t) - f(t) = t^2 + e^t$$

Solution:
a) $C_1 e^{-t} + t^2 - 2t + 3$.
b) $C_1 \cos(3t) + C_2 \sin(3t) + t^2/9 + 7/81$.
c) $C_1 e^{-t} + c_2 te^{-t} + t^2 - 4t + 6$
d) $C_1 e^t + c_2 e^{-1t} + te^t/2 - t^2 - 2$.

Problem 11) (10 points)

We analyze the following nonlinear dynamical system

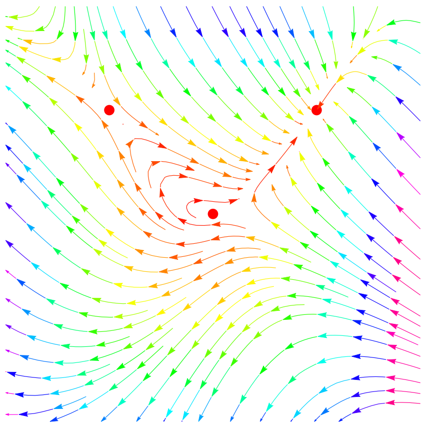
$$\begin{aligned} \frac{d}{dt}x &= y - x^2 \\ \frac{d}{dt}y &= x^2 - y^2 \end{aligned}$$

a) (2 points) Find the equations of the null-clines.

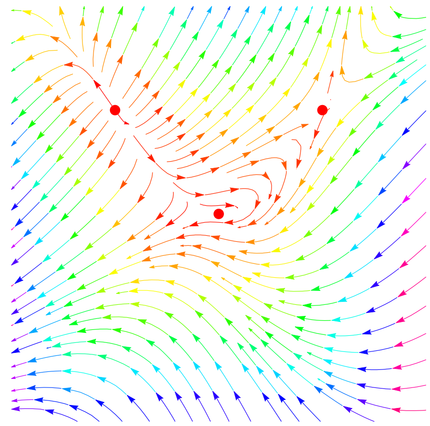
b) (2 points) Find all the equilibrium points.

c) (3 points) Analyze the stability of the equilibrium points.

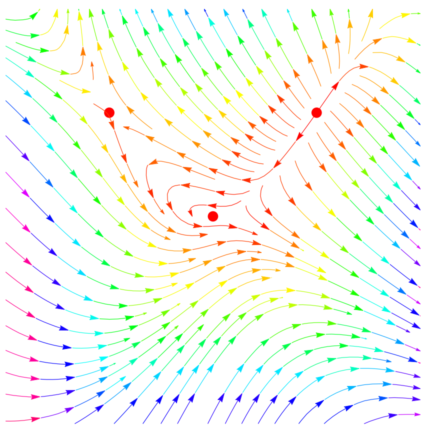
d) (3 points) Which of the four phase portraits A,B,C,D below belongs to the above system? Make sure that also here, you justify your answer, as always.



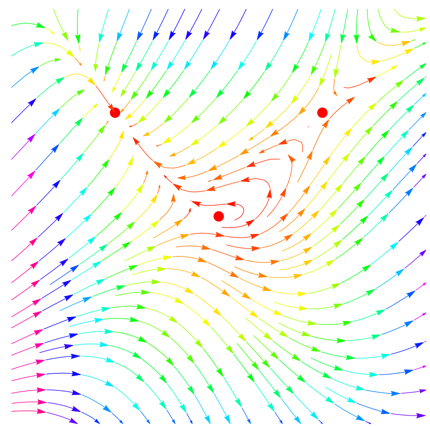
A



B



C



D

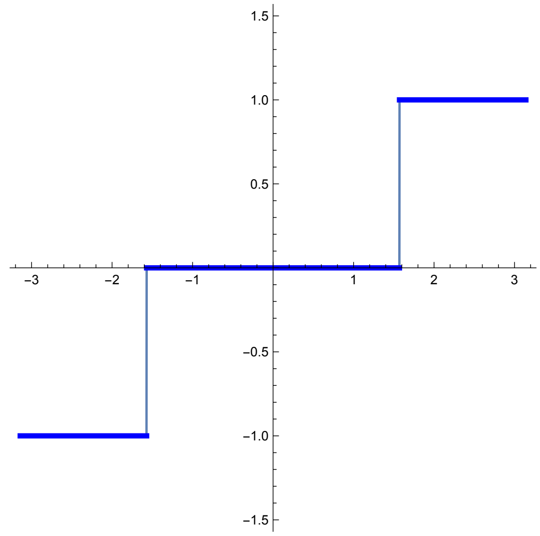
Solution:
a) The nullclines are $y = x^2$ and $y^2 = x^2$.
b) The equilibrium points are $(0,0), (1,1), (-1,1)$. c) The Jacobean matrix is $J = \begin{bmatrix} -2x & 1 \\ 2x & -2y \end{bmatrix}$. at the critical points we have $J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $J(1,1) = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}$ and $J(-1,1) = \begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix}$. The points $(0,0)$ and $(-1,1)$ are not stable. The point $(1,1)$ is stable.
d) Comparing the stability at the three points shows that A is the only choice.

Problem 12) (10 points)

a) (7 points) Find the **Fourier series** of the function

$$f(x) = \begin{cases} 1 & \frac{\pi}{2} < x < \pi \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & -\pi < x < -\frac{\pi}{2} \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



b) (3 points) Given the Fourier coefficients a_0, a_n, b_n in the previous problem, find the sum

$$\sum_{n=0}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 .$$

Solution:
a) The function is odd. It therefore has a sin series. we have

$$b_n = \frac{2}{\pi} \int_{\pi/2}^{\pi} \sin(nx) \, dx = \frac{2}{n\pi}(\cos(n\pi/2) - \cos(n\pi)) .$$

The series is $\sum_{n=1}^{\infty} b_n \sin(nx)$.
b) The Parcival identity tells that the sum is $||f||^2 = \frac{2}{\pi} \int_{\pi/2}^{\pi} 1^2 \, dx = 1$.

Problem 13) (10 points)

a) (5 points) The partial differential equation

$$u_t = u_{xx} + u$$

is a modification of the **heat equation** on the interval $[0, \pi]$. Solve it with the initial condition

$$u(x, 0) = \sin(x) + 4\sin(7x) + 2\sin(13x) .$$

b) (5 points) The partial differential equation

$$u_{tt} = u_{xx} + u$$

is a modification of the **wave equation** on the interval $[0, \pi]$. Solve it with the initial conditions

$$u(x, 0) = 2\sin(5x), u_t(x, 0) = 3\sin(5x) + 6\sin(8x) .$$

P.S. As usual, we only consider sin-Fourier expansions in x.

Solution:
a) The eigenvalues of the operator $Tf = f_{xx} + u$ on the right hand side are $\lambda = -n^2 + 1$ for the eigenvector $\sin(nx)$. This reduces the heat equation to $u' = -\lambda u$ and a factor $e^{-\lambda t}$ in each case.
The solution is

$$u(x,t) = \sin(x)e^{(1-1^2)t} + 4\sin(7x)e^{(1-7^2)t} + 2\sin(13x)e^{(1-13^2)t} .$$

b) We have the same eigenvalue but since we want to reduce the equation $u_{tt} = \lambda u$ to a harmonic oscillator, we have $u_{tt} + c^2 u = 0$ and $c = \sqrt{-\lambda}$. For the initial position component we have to include $\cos(ct)$ and for the initial velocity factor, we have to include $\sin(ct)/c$. The solution is

$$u(x,t) = 2\sin(5x)\cos(\sqrt{5^2-1}t) + 3\sin(5x)\frac{\sin(\sqrt{5^2-1}t)}{\sqrt{5^2-1}} + 6\sin(8x)\frac{\sin(\sqrt{8^2-1}t)}{\sqrt{8^2-1}} .$$

Problem 14) (10 points)

a) (7 points) The **pseudo determinant** of a matrix is the product of the nonzero eigenvalues of a matrix. Find the Pseudo determinant of

$$\begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \end{bmatrix} .$$

b) (3 points) We know that $\det(AB) = \mathbf{det}(A)\mathbf{det}(B)$. This identity is no more true for pseudo determinants. Find a counter example. Hint: You can find diagonal examples.

Solution:
a) The eigenvalues are 0 with multiplicity 9. The largest eigenvalue is 90, the trace.
b) Take $A = Diag(2,1,0)$ and $B = Diag(0,1,2)$. Then $AB = Diag(0,1,0)$ has pseudo determinant 1, while A, B both have pseudo determinant 2.