MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F Any homogeneous system of linear equations $Ax = 0$ is consistent.

Solution:
Consistent means that there is at least one a solution. And yes, there is the solution $x = 0$.

2) T F If P is the matrix of a projection onto a line, then all coefficients of the matrix satisfy $|P_{ij}| \leq 1$.

Solution:
This follows from the explicit formula $P_{ij} = u_i u_j$ where u is a unit vector in the line onto which one projects.

3) T F The transformation $f(A) = AA^T - 1$ is linear on the 4-dimensional space X of all 2×2 matrices.

Solution:
It does not map the zero matrix to the zero matrix for example.

4) T F If a smooth 2π periodic function f has a sin-Fourier expansion then its derivative f' has a cos-expansion.

Solution:
Differentiate the series: $d/dt \sum_n b_n \sin(nx) = \sum_n n b_n cos(nx)$.

5) T F The characteristic polynomials of two $n \times n$ matrices A, B satisfy $f_A(\lambda) f_B(\lambda) = f_{AB}(\lambda)$.

Solution:
The formula does not even hold in one dimensions. This formula holds for partitioned matrices with matrices A, B in the diagonal and zero matrices in the side diagonal.

6) T F The function $f(t) = \cos(10t)$ is an eigenfunction of the linear operator $T = D^4$, where $Df = f'$ is the differentiation operator on $C^\infty(\mathbf{R})$.

Solution:
Yes, it is an eigenfunction to the eigenvalue 10^4.

7) T F The matrix $A^2 + (A^2)^T$ is diagonalizable, if A is a $n \times n$ matrix.

Solution:
Yes, the matrix $A + A^T$ is symmetric and so diagonalizable.

8) T F The initial value problem $f''(x)+f'(x)+f(x) = \cos(x)$, $f(0) = 0$ has exactly one solution.

Solution:
The solution space is one dimensional, and has so many solutions.

9) T F The transformation $T(f)(x) = f(\sin(x))$ is a linear transformation on the space $X = C^\infty(\mathbf{R})$ of smooth functions on the real line.

Solution:
We check three conditions: $T(0) = 0, T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f)$ as well as the condition that T maps the space into itself.

10) T F The space of smooth functions $f(x, t)$ of two variables which satisfy the partial differential equation $f_{tt} - f_{xx} = f$ is a linear space.

Solution:
Yes, if we add two functions which satisfy this differential equation, then the sum also satisfies this differential equation. Also a scaled function satisfies the differential equation. And the zero function also satisfies this differential equation.

11) T F If A is 6×6 matrix of rank 5, then it has an eigenvalue 0.

Solution:
Having an eigenvalue 0 is equivalent to have a nontrivial kernel.

12) T F The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has the $\mathcal{B} = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\}$-coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:
No, $[1, 0] = [1, 2] - [0, 2]$.

13) T **F** If all the geometric multiplicities of a matrix are equal to the algebraic multiplicities, then the matrix is symmetric.

Solution:
It can be not symmetric but all eigenvalues are different for example.

14) T **F** All points on the nullclines of a differential equation $\dot{x} = f(x, y), \dot{y} = g(x, y)$ consist of equilibrium points.

Solution:
Only the place where an x-nullclines intersects a y-nullcline consists of equilibrium points.

15) **T** F If $z = (a + ib)$ is a nonzero complex number, then $1/z = (a - ib)/(a^2 + b^2)$.

Solution:
Yes, this is the definition.

16) T **F** On the space of 2×2 matrices, the trace tr satisfies $\mathrm{tr}(A^2) = \mathrm{tr}(A)^2$.

Solution:
It is already not true for diagonal matrices.

17) **T** F The QR decomposition of an orthogonal matrix A is $A = QR$, where $Q = A$ and $R = 1_n$.

Solution:
It is already the QR decomposition.

18) **T** F For any real numbers a, b, there exists a 2×2 matrix such that a is the trace and b is the determinant.

Solution:
The matrix $A = \begin{bmatrix} a & -1 \\ b & 0 \end{bmatrix}$ has trace a and determinant b.

19) **T** F The dynamical system $x(t+1) = (1/2)x(t) + (1/3)x(t-1)$ has the property that $x(t) \to 0$ for all initial conditions $(x(0), x(1))$.

Solution:
Write it as a discrete dynamical system, which corresponds to iterating the matrix $\begin{bmatrix} 1/2 & 1/3 \\ 1 & 0 \end{bmatrix}$. This matrix has eigenvalues $(3 \pm \sqrt{57})/12$, which are both in modulus smaller than 1.

20) T | F — For a symmetric matrix A, the kernel of A is perpendicular to the image of A.

Solution:
We know that the kernel of A^T is perpendicular to the image of A. For symmetric matrices, the kernel of A is perpendicular to the image of A.

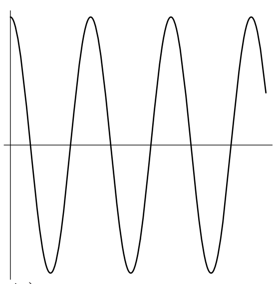
Problem 2) (10 points)

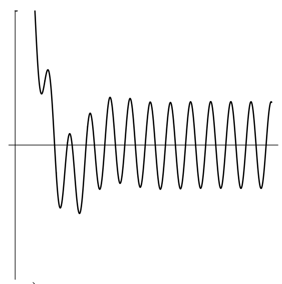
No justifications are needed in this problem. Match the equations with the solution graphs $f(x)$. Enter A,B,C in the right order here:

$f''(x) + f'(x)/5 + f(x) = 0, f(0) = 1, f'(0) = 0$

$f''(x) + f'(x) + f(x) = 5\sin(4x), f(0) = 1, f'(0) = 0$

$f''(x) + f(x) = 0, f(0) = 1, f'(0) = 0.$


A)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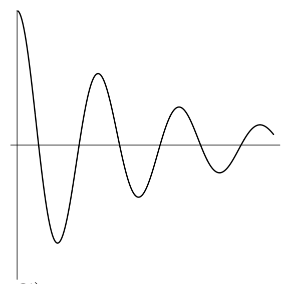
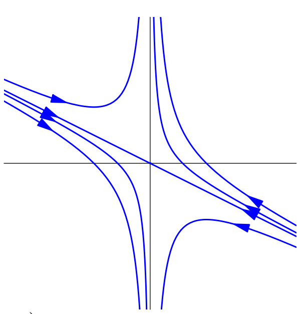

C)

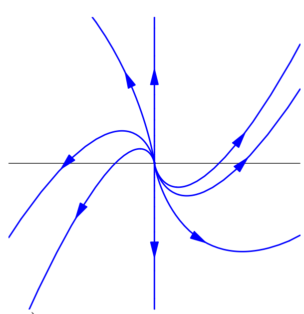
Match the phase portraits for the continuous dynamical systems. Enter D,E,F in the right order here:

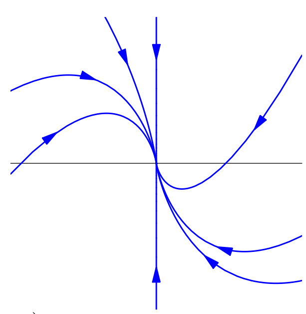
$\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} \vec{x}$

$\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}$

$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \vec{x}$


D)


E)


F)

Solution:
C,B,A,F,D,E.

Problem 3) (10 points)

No justifications are needed in this problem. Fill in a choice of the letters $A-J$ in each of the boxes. No letter will appear twice.

	$\dim(\ker(A)) + \dim(\mathrm{im(A)}) =$ is the number of columns of A.
	If p is a polynomial of degree n, we can write $p(D) = a(D-\lambda_1)(D-\lambda_2)...(D-\lambda_n)$.
	If f is odd, then $\frac{2}{\pi}\int_0^\pi f(x)^2\, dx = \sum_{n=1}^\infty (\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\, dx)^2$.
	If $\mathrm{tr}(A) < 0$ and $\det(A) > 0$, then the two dimensional dynamical system is stable.
	The dimension of $\{ f \in C^\infty(\mathbf{R}) \mid p(D)f = 0 \}$ is the degree of the polynomial p.
	If $\int_{-\pi}^\pi f(x)g(x)\, dx = 0$, then $\int_{-\pi}^\pi f(x)^2\, dx + \int_{-\pi}^\pi g(x)^2\, dx = \int_{-\pi}^\pi (f(x)+g(x))^2\, dx$.
	$\cos(\theta) + i\sin(\theta) = e^{i\theta}$.

A) Theorem of Pythagoras for inner products.
B) Fundamental theorem of algebra.
C) Stability for linear continuous dynamical systems in the plane.
D) Percival identity.
E) Rank nullity theorem.
F) Euler's formula.
G) sin-Fourier expansion.
H) Stability criterion for linear discrete dynamical systems in the plane.
I) Parameterization of homogeneous solution space.
J) The Laplace expansion.

Solution:
E,B,D,C,I,A,F

Problem 4) (10 points)

Find all the solutions of the system of linear equations:

$$\begin{array}{ccccc} x & +y & +z & +v & = 4 \\ & 2y & +z & +v & = 4 \\ & & 3z & +v & = 4 \\ x & +3y & +2z & +2v & = 8 \end{array}$$

Solution:
We row reduce the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 2 & 2 & 8 \end{array}\right]$$

which leads to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/3 & 4/3 \\ 0 & 1 & 0 & 1/3 & 4/3 \\ 0 & 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] .$$

The system is consistent and has the solution

$$\begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix} .$$

The general solution can be written in many different ways. An other possibility without fractions is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} .$$

Problem 5) (10 points)

We consider the matrix

$$A = \begin{bmatrix} -6 & 1 & 1 & 1 & 1 & 1 \\ 1 & -6 & 1 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 1 & 1 \\ 1 & 1 & 1 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & -6 & 1 \\ 1 & 1 & 1 & 1 & 1 & -6 \end{bmatrix} .$$

a) (2 points) Find all the eigenvalues of $B = A + 7I_6$.
b) (3 points) Find all the eigenvectors of B.
c) (2 points) Find all the eigenvalues of A.
d) (3 points) Decide about the stability for the continuous dynamical system.

$$\dot{x} = Ax .$$

As usual, document all your reasoning.

Solution:
a) The eigenvalues of B are 0 (with algebraic multiplicity 5) and 6 with algebraic multiplicity 1.
b) Five of the eigenvectors span the kernel of A and are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} .$$

The eigenvector to the eigenvalue 6 is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

c) The eigenvalues of A are the eigenvalues of B minus 7. They are -1 (with multiplicity 1) and -7 (with multiplicity 5).
d) All eigenvalues are negative. The origin is stable.

Problem 6) (10 points)

a) (5 points) Find the 3×3 matrix for the linear transformation which reflects at the plane $x = z$ in three dimensional space.

b) (5 points) What is the 3×3 matrix which rotates by 180 degrees around the line perpendicular to the plane $x = z$ in three dimensional space?

Solution:
a) One possibility is to look at the image of the standard basis. Like this, we directly get the columns of the matrix. Because $e_1 \to e_3, e_3 \to e_1, e_2 \to e_2$, we have

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

An other possibility is to describe the map in a natural basis first. This would be

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

In this basis, the reflection is

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The change of variable matrix is

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} .$$

We have $A = SBS^{-1}$.
b) The rotation sends e_2 to $-e_2$, $e_1 \to -e_3$ and $e_3 \to -e_1$. The matrix is

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} .$$

The problem can be solved also with the same basis. In the adapted basis, the matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

The matrix we are looking for is $A = SBS^{-1}$.

Problem 7) (10 points)

Find the least square solution of the following system of linear equations with the unknown variables x, y:

$$\begin{array}{rcl} x+y &=& 1 \\ x-y &=& 2 \\ 2x+y &=& 1 \end{array}$$

Solution:
The least square solution is $x_* = (A^TA)^{-1}A^Tb$. We have $A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}/14$. $A^Tb = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Together, $x_* = \begin{bmatrix} 15 \\ -10 \end{bmatrix}/14$.

Problem 8) (10 points)

The difference equation

$$x_{n+1} = 6x_n - 5x_{n-1}$$

can be written as a discrete dynamical system

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} .$$

Starting with $x_0 = 0, x_1 = 4$, find an explicit formula for x_n in terms of n.

Solution:
The eigenvalues of A are 5 and 1 with eigenvectors $w_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We can write the initial condition $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The solution is

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = 1 \cdot 5^n \begin{bmatrix} 5 \\ 1 \end{bmatrix} - 1 \cdot 1^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

This means $x_n = 5^n - 1$.

Problem 9) (10 points)

Find the determinant of the following 36×36 matrix:

$$\begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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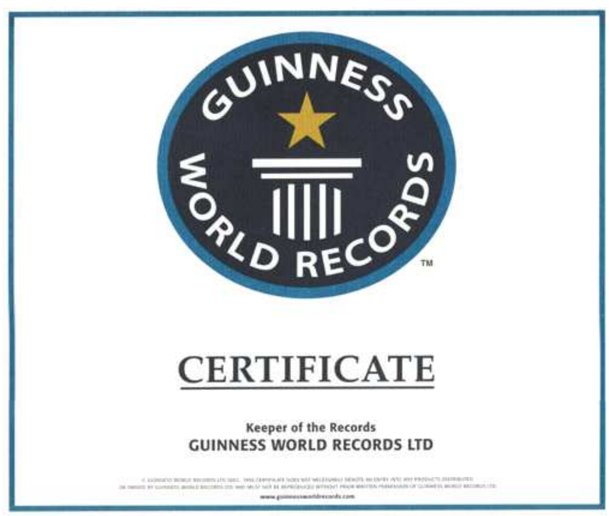
\end{bmatrix}.$$

Make sure you document all your thinking and facts you use about determinants. A detailed protocol can also help to get partial credit.

With this problem we got into the book of Guinness records as the largest matrix which has ever occurred in a linear algebra test.



Solution:
First partition the matrix into 4 large 18x18 matrices. The determinant is the product of the two matrices in the diagonal. Each of these two identical 18×18 matrices A, B is again a partitioned matrix, containing 9 matrices of size 2×2 each having determinant -1. The determinant of A and B are both -1. The determinant is $(-1) \cdot (-1) = 1$.

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (2 points) $f'(t) = t$.
b) (2 points) $f'(t) + f(t) = t^2$.
c) (2 points) $f''(t) - f(t) = t$
d) (2 points) $f''(t) + 9f(t) = t$
e) (2 points) $f''(t) - 2f'(t) + f(t) = t$

Solution:
a) $t^2/2 + C$,
b) $C_1 e^{-t} + t^2 - 2t + 2$.
c) $C_1 e^{-t} + C_2 e^t - t$.
d) $C_1 \cos(3t) + C_2 \sin(3t) + t/9$.
e) $C_1 e^t + C_2 t e^t + t + 2$.

Problem 11) (10 points)

We analyze the following nonlinear system of differential equations:

$$\begin{aligned} \dot{x} &= y - xy \\ \dot{y} &= x + xy \end{aligned}$$

a) (3 points) Find the nullclines.
b) (3 points) Find all equilibrium points.
c) (4 points) Determine the stability of these equilibrium points.

Solution:
a) The x-nullclines are the lines $y = 0$, or $x = 1$.
The y-nullclines are the lines $x = 0$, or $y = -1$.
b) There are two equilibrium points: $(0,0), (1,-1)$.
c) The Jacobian matrix is

$$J = \begin{bmatrix} -y & 1-x \\ 1+y & x \end{bmatrix} .$$

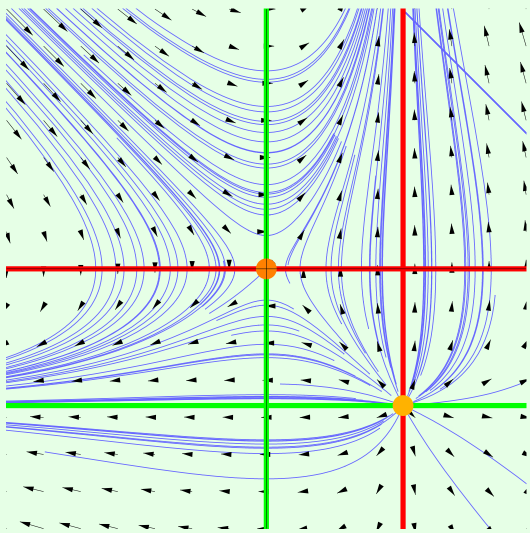
At the point $(0,0)$, the Jacobian is

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has eigenvalues 1 and -1. This is an unstable point.
At the point $(1,-1)$, we have

$$J(1,-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has two eigenvalues 1. Also this is an unstable point.



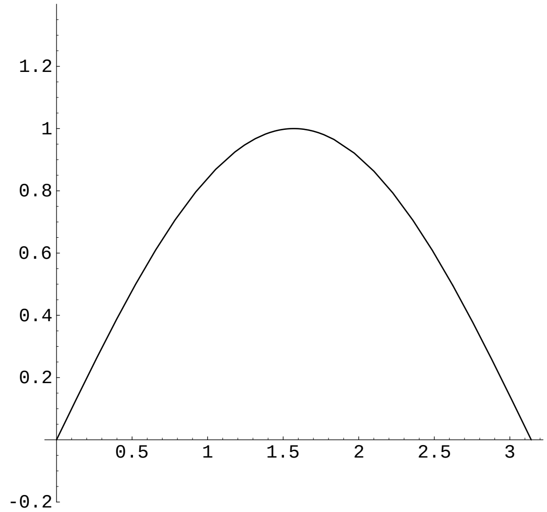
Problem 12) (10 points)

Find the solution of the wave equation $f_{tt} = 16 f_{xx}$ with initial string position

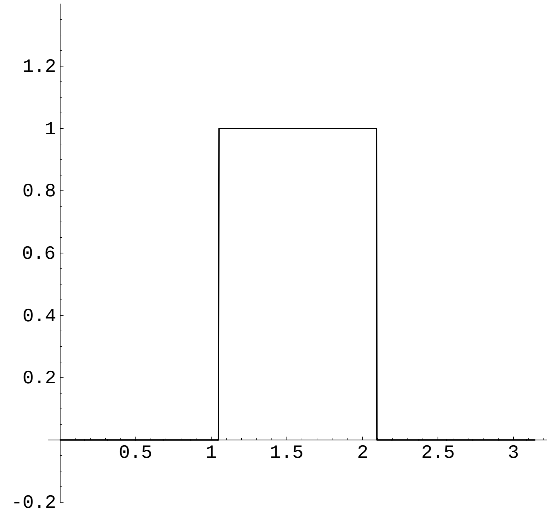
$$f(x,0) = \sin(x)$$

and initial string velocity

$$f_t(x,0) = \begin{cases} 1, & \pi/3 \leq x \leq 2\pi/3 \\ 0, & \text{else} \end{cases} .$$



Initial wave position on $[0, \pi]$.



Initial wave velocity on $[0, \pi]$.

Solution:
The Fourier decomposition of the initial wave position is $\sin(x)$. We do not need to develop this further because a trigonometric polynomial is already its Fourier expansion. The Fourier decomposion of the initial wave velocity is $\sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$b_n = \frac{2}{\pi} \int_{\pi/3}^{2\pi/3} \sin(nx)\, dx = \frac{2}{n\pi}[\cos(n\pi/3) - \cos(2n\pi/3)] \ .$$

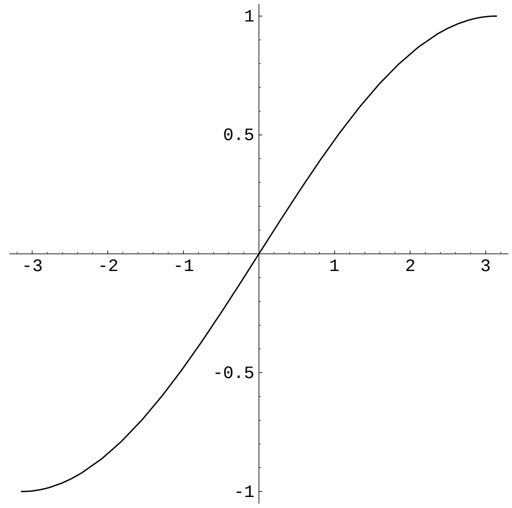
The solution of the wave equation is

$$f(x,t) = \cos(4t)\sin(x) + \sum_n b_n \frac{\sin(4nt)}{4n} \sin(nx) \ .$$

Problem 13) (10 points)

a) (6 points) Oliver just bought a new digital piano. One of the wave forms programmed into the piano is the function $\sin(x/2)$ on $[-\pi, \pi]$. Find the Fourier series of this function.

Hint. You can use the formula $2\sin(nx)\sin(mx) = \cos(nx-mx)-\cos(nx+mx)$ and identities like $\sin(a + \pi/2) = \cos(a), \sin(a - \pi/2) = -\cos(a)$.



b) (4 points) What is the value of

$$\sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2} \ ?$$

Solution:
a) The function is odd. It has a sin-expansion. We have

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin(x/2) \sin(nx) \, dx \\ &= \frac{2}{2\pi} \int_0^{\pi} \cos((n-1/2)x) - \cos((n+1/2)x) \, dx \\ &= \frac{1}{\pi} \frac{\sin((n-1/2)x)}{n-1/2} \Big|_0^{\pi} - \frac{\sin((n+1/2)x)}{n+1/2} \Big|_0^{\pi} \\ &= \frac{-1}{\pi} \frac{\cos(n\pi)}{n-1/2} + \frac{\cos(n\pi)}{n+1/2} \\ &= \frac{8n\cos(n\pi)}{\pi(1-4n^2)} = \frac{8n(-1)^n}{\pi(1-4n^2)} \end{aligned}$$

The Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{8n(-1)^n}{\pi(1-4n^2)} \sin(nx) \ .$$

b) We use Percevals identity $||f||^2 = \sum_{n=1}^{\infty} b_n^2$. The left hand side of the identity is

$$||f||^2 = \frac{2}{\pi} \int_0^{\pi} \sin^2(x/2) \, dx = 1 \ .$$

The right hand side is $\sum_{n=1}^{\infty} b_n^2 = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2}$. The result is $\boxed{\pi^2/64}$.

Problem 14) (10 points)

In this problem, we verify

$$1 + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} .$$

It is the value of the famous zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at the point $s = 4$.
We have seen in the handout that $f(x) = x$ has the Fourier expansion

$$x = \sum_{n=1}^{\infty} 2(-1)^{(n+1)} \frac{\sin(nx)}{n} \tag{1}$$

a) You do not have to reverify this here. But we want to know, how the Fourier coefficients $b_n = 2(-1)^{(n+1)}/n$ were obtained in that expansion. Write down the formulas for the Fourier coefficients and especially answer why there are no a_n terms.

b) By taking the anti-derivative of the above formula on both sides, we get

$$g(x) = \frac{x^2}{2} = \sum_{n=1}^{\infty} 2(-1)^n \frac{\cos(nx)}{n^2} + a_0 \frac{1}{\sqrt{2}} .$$

Find a_0, the zero'th Fourier coefficient of the function $\frac{x^2}{2}$.

c) Compute the length $||g||^2 = \langle g, g \rangle$ and deduce from this

$$4 \sum_{n=1}^{\infty} \frac{1}{n^4} = (\frac{\pi^4}{10} - \frac{\pi^4}{18}) = 2\pi^4/45$$

which implies $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Solution:
This problem had been solved in the review lecture.
a) Since the function was odd, it had a sin expansion. We have

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) \, dx .$$

b) When taking the antiderivative, the integration constant was not determined yet, and a_0 fixes that. We have

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (x^2/2) 1/\sqrt{2} \, dx = \pi^2/(3\sqrt{2})$$

Its square is $a_0^2 = \frac{\pi^4}{18}$).
c) The length of the function is

$$\frac{2}{\pi} \int_0^{\pi} (x^2/2)^2 \, dx = \frac{\pi^4}{10} .$$

Putting things together gives the value of the zeta function $\zeta(4)$.